*Notes: some of these problems (denoted *) extend beyond what was convered in the lectures and will require some extra reading.*

Broken Symmetry

1. The Landau theory of a ferromagnet in a magnetic field H implies that the free energy is given by
$$F(M) = F_0 + a_0(T - T_C)M^2 + bM^4 - \mu_0 MH \tag{1}$$
where a_0 and b are positive constants.
(a) By evaluating $\partial F/\partial M$, and setting it equal to zero, show that for no applied field ($H = 0$) the magnetization M is zero above T_C and is proportional to $(T_C - T)^{1/2}$ below T_C.
(b) Show also that at $T = T_C$, $M \propto H^{1/3}$.
(c) Show that the form of the free energy also implies that
$$M^2 = u + v\frac{H}{M} \tag{2}$$
where u and v are constants that you should determine. By sketching M^2 against H/M for T just above T_C, just below T_C, and exactly at T_C, show how this method can be used to determine T_C. This idea is the basis of the so-called *Arrott plot* which is a plot of M^2 against H/M which is used experimentally to locate T_C from $M(H)$ data measured at different temperatures.

2. * *This problem requires you to read a description of Widom's hypothesis.*
(a) Using Widom's hypothesis in the form $f(t,h) = t^{\frac{d}{y_t}}\tilde{f}(h/t^{\frac{y_h}{y_t}})$, show that critical parameters are given (in terms of the scaling parameters y_h and y_t and the dimensionality d) as follows
$$\alpha = 2 - \tfrac{d}{y_t}, \quad \beta = \tfrac{d-y_h}{y_t}, \quad \gamma = \tfrac{2y_h-d}{y_t}. \tag{3}$$
(b) We now redefine the scaling function slightly so that it reads $f(t,h) = h^{\frac{d}{y_h}}\tilde{g}(h/t^{\frac{y_h}{y_t}})$ where $g(z) = z^{-\frac{d}{y_h}}f(z)$. Use this form of the function to show that $\delta = \frac{y_h}{d-y_h}$.
(c) From these results prove the following relations:
$$\begin{aligned} \alpha + 2\beta + \gamma = 2 &\quad \text{Rushbrooke's law,} \\ \alpha + \beta(\delta+1) = 2 &\quad \text{Griffith's law.} \end{aligned} \tag{4}$$
(d) Assuming the correlation function behaves as $G_c(x,t) = f\left(xt^{\frac{2-\alpha}{d}}\right)/x^{d-2+\eta}$, argue that we require $\nu d = 2 - \alpha$ (known as Josephson's law) and use this to show $\nu = \frac{1}{y_t}$.
(e) Finally use Fisher's law $(2-\eta)\nu = \gamma$ to find η.

Landau-Fermi liquids

1. What is the average energy for an electron in the (non-interacting) Fermi gas?

2. *This argument about scattering in a Fermi liquid appears in many books.*
Consider an electron with energy $E_1 \geq E_F$ scattering with an electron with energy $E_2 \leq E_F$ at $T = 0$. In order for this to occur we must have final electron states $E_3 \geq E_F$ and $E_4 \geq E_F$.
(a) Show that this implies that the lifetime of an electron with $E_1 = E_F$ is infinite.
(b) If E_1 is a little different to E_F, why does the scatter scattering rate vary as $(E_1 - E_F)^2$?
(c) For $T \neq 0$ argue that we expect a scattering rate $\frac{1}{\tau} = a(E - E_F)^2 + b(k_B T)^2$, where a and b are constants.

Simple Harmonic Oscillator

The simple harmonic oscillator problem is described by the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2\hat{x}^2}{2}$ and the commutation relation $[x, p] = \mathrm{i}\hbar$. Consider the creation ($\hat{a}^\dagger$) and annihilation ($\hat{a}$) operators we defined in the lecture.

1. Show that $[\hat{a}, \hat{a}] = 0$, $[\hat{a}^\dagger, \hat{a}^\dagger] = 0$, $[\hat{a}, \hat{a}^\dagger] = 1$ and $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$.
2. Consider a perturbation to this Hamiltonian $\beta\hat{x}^3 + \gamma\hat{x}^4$ where β and γ are small. By writing the perturbation in terms of creation and annihilation operators of the original Hamiltonian, show that the first-order shift in the ground-state energy of the system, due to these anharmonic parts, is given by
$$\Delta E = \frac{3}{4}\gamma\left(\frac{\hbar}{m\omega}\right)^2.$$
3. (a) Show that the transformation $\hat{b} = u\hat{a} + v\hat{a}^\dagger$ and $\hat{b}^\dagger = u\hat{a}^\dagger + v\hat{a}$ (with u and v real), preserves the commutation relations, as long as $u^2 - v^2 = 1$.

   * (b) Using the results of (a), diagonalize the Hamiltonian
$$H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \frac{\Delta}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}\right), \tag{5}$$
   by transforming it into the form $H = \hbar\varepsilon\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right)$ and find ε. This is an example of a Bogoliubov transformation and is a useful trick to diagonalize a Hamiltonian. *Hint: If you have a problem with the algebra, see J.F. Annett, Superconductivity, Superfluids and Condensates for some help.*

Quantum fields

1. (a) Show that $\mathrm{i}\frac{\partial\hat{U}}{\partial t} = \hat{H}\hat{U}$, where $\hat{U}$ is shorthand for the time evolution operator $\hat{U}(t, 0) = e^{-\mathrm{i}\hat{H}t}$.
   (b) By differentiating $\hat{O}_{\mathrm{H}}(t) = e^{\mathrm{i}\hat{H}t}\hat{O}e^{-\mathrm{i}\hat{H}t}$, derive Heisenberg's equation of motion.
2. * (a) $\hat{V}(\mathbf{a})$ is a translation operator with the property $\hat{V}(\mathbf{a})|\mathbf{x}\rangle = |\mathbf{x} + \mathbf{a}\rangle$. Show that, for an operator valued field $\hat{\phi}(\mathbf{x})$, we have $\hat{V}^\dagger(\mathbf{a})\hat{\phi}(\mathbf{x})\hat{V}(\mathbf{a}) = \hat{\phi}(\mathbf{x} - \mathbf{a})$.

   * (b) By considering an infinitesimal translation show that an explicit form for the translation operator is $\hat{V}(\mathbf{a}) = e^{-\mathrm{i}\hat{\mathbf{p}}\cdot\mathbf{a}}$. *Hint: If you find you have the wrong sign in your exponential, consider the difference between translating a particle and allowing it to evolve.*

Examples of second quantization

1. An electron system has three momentum states, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, and is described by a Hamiltonian
$$\hat{H} = E_0\sum_{\mathbf{p}}\hat{d}^\dagger_{\mathbf{p}}\hat{d}_{\mathbf{p}} - \frac{V}{2}\sum_{\mathbf{pk}}\hat{d}^\dagger_{\mathbf{k}}\hat{d}_{\mathbf{p}}. \tag{6}$$
   States are expressed using a basis $|n_{\mathbf{p}_1}n_{\mathbf{p}_2}n_{\mathbf{p}_3}\rangle$ and if we put a single electron into the system then its state may be written $|\psi\rangle = a|100\rangle + b|010\rangle + c|001\rangle$.
   Show that the Hamiltonian takes the form
$$\hat{H} = \left[E_0\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} - \frac{V}{2}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\right]. \tag{7}$$
   Find the energy eigenvalues and the corresponding eigenstates.

2. The nearest neighbour Hubbard model Hamiltonian may be written

$$\hat{H} = -t \sum_{\langle ij \rangle} (\hat{c}^{\dagger}_{i\sigma} \hat{c}_{j\sigma} + \hat{c}^{\dagger}_{j\sigma} \hat{c}_{i\sigma}) + U \sum_{i} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}, \tag{8}$$

where the first sum is over unique nearest neighbours. Consider a system with two possible sites for electrons.

(a) Put a single electron in the system. Using a basis $|\uparrow, 0\rangle$ and $|0, \uparrow\rangle$ show that the Hamitonian is given by

$$\hat{H} = \begin{pmatrix} 0 & -t \\ -t & 0 \end{pmatrix}. \tag{9}$$

Find the energy eigenvalues and eigenstates.

(b) Now put a second electron into the system with opposite spin to the first. Now using the basis states $|\uparrow\downarrow, 0\rangle$; $|\uparrow, \downarrow\rangle$; $|\downarrow, \uparrow\rangle$; and $|0, \downarrow\uparrow\rangle$, show that the Hamiltonian becomes

$$\hat{H} = \begin{pmatrix} U & -t & -t & 0 \\ -t & 0 & 0 & -t \\ -t & 0 & 0 & -t \\ 0 & -t & -t & U \end{pmatrix}. \tag{10}$$

Diagonalize this to obtain the eigenstates and energy eigenvalues. *Hint: There's no shame in using a computer if you like!*

Propagators and perturbation theory

1. Show that the single particle propagator $G = \langle x, t_x | y, t_y \rangle$ may be written

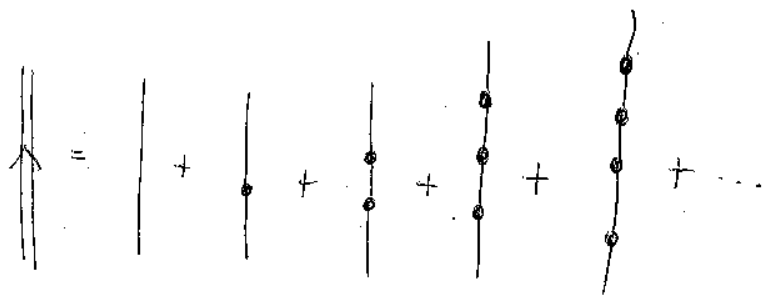
$$G = \sum_{p} \phi_p(x) \phi^*_p(y) e^{-\mathrm{i}E_p(t_x - t_y)}. \tag{11}$$

2. For non-relativistic, free particles in one dimension, show that the propagator is given by

$$G = \sqrt{\frac{m}{2\pi\mathrm{i}(t_x - t_y)}} e^{\frac{\mathrm{i}m(x-y)^2}{2(t_x - t_y)}}. \tag{12}$$

3. Consider the Lagrangian density $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2$. We're going to treat the mass term as a perturbation by splitting the theory into a free part $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2$ and an interacting part $\mathcal{L}_{\text{int}} = -\frac{m^2}{2}\phi^2$. The free propagator is given, in momentum space, by $G_0(p) = \frac{\mathrm{i}}{p^2}$.

In order to see how the perturbation modifies the propagator consider the infinite sum of diagrams in the figure.



If each interaction blob contributes a factor $-\mathrm{i}m^2$ show that the full propagator is given by

$$G = \frac{\mathrm{i}}{p^2 - m^2}. \tag{13}$$