

# Lecture 1

①

① Introduction: The problems with Quantum mechanics

1.1 Relativity: Let's try to make a relativistic QM

$$\hat{p} \rightarrow -i\hbar \nabla \quad E \rightarrow i\hbar \frac{\partial}{\partial t}$$

$E^2 = p^2 + m^2$  in relativistic physics [we set  $c=1$ ]

$$-t^2 \frac{\partial^2}{\partial t^2} = -\nabla^2 + m^2 \quad [\text{we also set } \hbar=1]$$

$$\left[ \frac{\partial^2}{\partial t^2} - \nabla^2 \right] + m^2 = 0$$

$$\left[ \nabla^2 + m^2 \right] \psi = 0 \quad \text{This is the 'Klein-Gordon' equation}$$

This seems okay, but there's a big problem. We'll substitute in a plane wave  $\psi = A e^{-iEt + ip \cdot x}$ . We find  $\nabla^2 \psi = (-E^2 + p^2) \psi$

The Klein-Gordon equation gives us

$$(-E^2 + p^2) \psi + m^2 \psi = 0$$

or  $E^2 = p^2 + m^2$ , which looks fine until we realize that this implies  $E = \pm \sqrt{p^2 + m^2}$ .

Energies come out negative (as well as positive). This is a disaster!

1.2 Many particle physics - the problem of  $n$  identical particles.

We can write down a 2 electron wavefunction, which must be asymmetric on exchange of particles

$$\psi(r_1, r_2) = \frac{1}{\sqrt{2}} [\phi_a(x_1) \phi_b(x_2) - \phi_b(x_1) \phi_a(x_2)]$$

What about for  $n$  particles?

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For  $n$  particles  $\Psi(r_1, \dots, r_n) \propto$

$$\begin{vmatrix} \phi_a(r_1) & \phi_b(r_1) & \cdots & \phi_\omega(r_1) \\ \phi_a(r_2) & \phi_b(r_2) & & \phi_\omega(r_2) \\ \vdots & & & \\ \phi_a(r_n) & \phi_b(r_n) & & \phi_\omega(r_n) \end{vmatrix}$$

Base  
Fermi

This is an awful & complicated business. There must be a better way.

1.3 The solution : It turns out that the solution to both of these problems is the same. We switch to a view of the world involving FIELDS. Fields are the heroes of the course!

In particular :

- (I) Every identical particle is an excitation of a field
- (II) Particles interact by exchanging 'virtual' particles
- (III) The particles are described by simple techniques based on the humble SHO.

In addition conservation laws are explained by symmetries of the fields

(4)

Quantum field theory is currently the best theory we have that describes the world around us. Various branches and applications include:

- Quantum electrodynamics (photons & electrons etc)
- Electroweak and Chromodynamics
- Critical phenomena & phase transitions
- Many body Condensed matter (metals, superconductors, LHe)
- Topological theories (Quantum Hall, solitons, Insulators)
- Soft matter problems (Melting, dynamics, defects etc)
- + More.

- Finally, one fundamental fact:

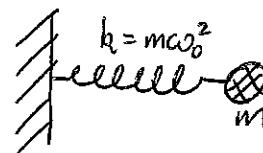
\*\* QFT is the way it is because it's the only way anyone knows of to reconcile conventional QM and special Relativity. \*\*

## 2 Second Quantization : Making things look like oscillators

2.1 No physics course is complete without a review of the SHO

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2$$

with  $[\hat{x}, \hat{p}] = i\hbar$



To solve this we define some new operators  $\hat{a}$  and  $\hat{a}^\dagger$

$$\hat{x} = \left(\frac{\hbar}{2m\omega_0}\right)^{\frac{1}{2}} (\hat{a} + \hat{a}^\dagger) \quad \hat{p} = -i\left(\frac{m\hbar\omega_0}{2}\right)^{\frac{1}{2}} (\hat{a} - \hat{a}^\dagger)$$

and  $[\hat{a}, \hat{a}^\dagger] = 1$

$$\begin{aligned} H &= -\frac{\hbar\omega_0}{4} (\hat{a}^2 + \hat{a}^{\dagger 2} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}) + \frac{\hbar\omega_0}{4} (\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \\ &= \frac{\hbar\omega_0}{2} (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) = \frac{\hbar\omega_0}{2} (2\hat{a}^\dagger\hat{a} + 1) \leftarrow \text{using } [\hat{a}, \hat{a}^\dagger] = 1 \end{aligned}$$

So we have  $\hat{H} = \hbar\omega_0 (\hat{a}^\dagger\hat{a} + \frac{1}{2})$  and we know that  $E = \hbar\omega_0 (n + \frac{1}{2})$  (6)

We interpret  $\hat{a}^\dagger\hat{a} = \hat{n}$   $\leftarrow$  an operator telling us the  $n^{\circ}$  of quanta in a state.  
We label states  $|n\rangle$ , such that

$$\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle.$$

What about the state defined by  $|m\rangle = \hat{a}^\dagger|n\rangle$  ?

We hit this with a number operator

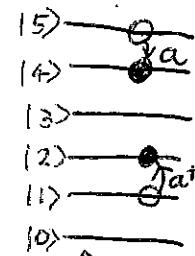
$$\begin{aligned} \hat{a}^\dagger\hat{a}|m\rangle &= \hat{a}^\dagger\hat{a}\hat{a}^\dagger|n\rangle \\ &= \hat{a}^\dagger(1 + \hat{a}\hat{a}^\dagger)|n\rangle \\ &= (1 + n)\hat{a}^\dagger|n\rangle \end{aligned}$$

Conclusion  $\hat{a}^\dagger$  increase the number of quanta by 1. In fact

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$\hat{a}^\dagger|n\rangle = (n+1)^{\frac{1}{2}}|n+1\rangle$$

$a^\dagger \rightarrow$  raising } operator  $\rightarrow$  creates 1 quantum  
creation } in the SHO

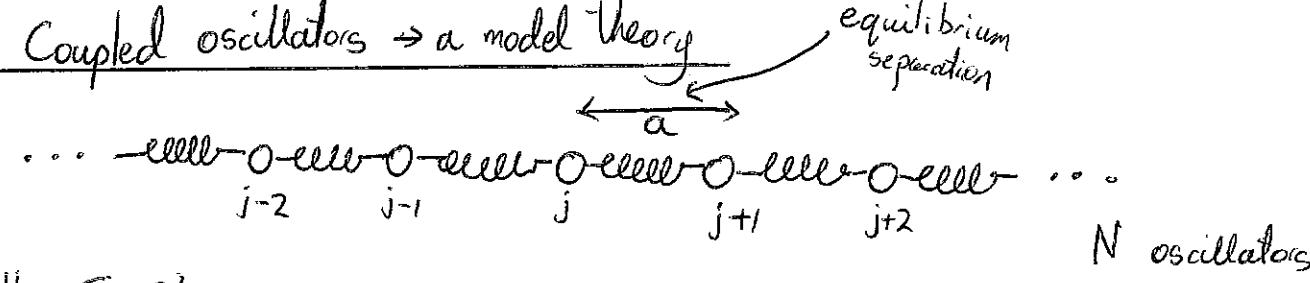


$a \rightarrow$  lowering } operator  $\rightarrow$  Destroys 1  
annihilation } quantum in the SHO

There must be a ground state such that  $a|0\rangle = 0$

This stuff is brilliant! We'd like to be able to use this machinery on other problems. If only...

## 2.2 Coupled oscillators $\rightarrow$ a model theory



$$H = \sum_j \frac{p_j^2}{2m} + \frac{1}{2} m \omega_0^2 (x_j - x_{j+1})^2$$

- 2 problems  $\rightarrow$  a) The oscillators are coupled  
 $\rightarrow$  b) we need to define creation/annihilation operators.

Problem (a) We decouple the oscillators using Fourier components

$$p_j = \frac{1}{\sqrt{N}} \sum_k p_k e^{ikja} \quad x_j = \frac{1}{\sqrt{N}} \sum_k x_k e^{-ikja}$$

$$\hat{H} = \frac{1}{2m} \sum_k \sum_q (p_k e^{ikja} p_q e^{iqja}) + \frac{mc\omega_0}{2N} \sum_j \sum_k \sum_q (x_k e^{ikja} - x_{k+1} e^{ik(j+1)a}) (x_q e^{iqja} - x_{q+1} e^{iq(j+1)a})$$

Consider the 1<sup>st</sup> term. We do the sum in 2 stages  $\nearrow$   
I Do the  $j$  sum  
II Do one of the momentum sums

Step (1): We use the fact that  $\sum_j e^{i\omega_j t} = N \delta_{w,0}$

$$\left( \begin{array}{l} \text{1st term} \\ \text{from H} \end{array} \right) = \frac{1}{2mN} \sum_{jkq} p_k p_q e^{i(k+q)t} a$$

$$= \frac{1}{2mN} \sum_{kq} p_k p_q N \delta_{k,-q}$$

Step (2): We need to do one of the momentum sums (which sets  $k = -q$ )

$$\left( \begin{array}{l} \text{1st term} \\ \text{from H} \end{array} \right) = \frac{1}{2m} \sum_q p_q p_{-q}$$

Next we'll use the same procedure on the 2<sup>nd</sup> term

$$\begin{aligned} \left( \begin{array}{l} \text{2nd term} \\ \text{from H} \end{array} \right) &= \frac{mc\omega_0^2}{2N} \sum_{jkq} x_k e^{ikj} (1 - e^{ika}) x_q e^{iqj} (1 - e^{iqa}) \\ [\text{STEP 1}] &= \frac{mc\omega_0^2}{2N} \sum_{jkq} x_k x_q (1 - e^{ika}) (1 - e^{iqa}) e^{i(k+q)ja} \\ &= \frac{mc\omega_0^2}{2N} \sum_{kq} x_k x_q (1 - e^{ika}) (1 - e^{iqa}) N \delta_{k,-q} \end{aligned}$$

Nothing to do but press on with step II & do the sum over k

$$\begin{aligned} \left( \begin{array}{l} \text{2nd term} \\ \text{from H} \end{array} \right) &= \frac{mc\omega_0^2}{2} \sum_q x_q x_{-q} (1 - e^{-iqa}) (1 - e^{iqa}) \\ &= \frac{mc\omega_0^2}{2} \sum_q x_q x_{-q} (2 - 2\cos qa) \end{aligned}$$

So we have the result

$$\boxed{\hat{H} = \sum_q \frac{1}{2m} p_q p_{-q} + \frac{1}{2} mc\omega_0^2 (2 - 2\cos qa) x_q x_{-q}}$$

and we define  $\omega_q^2 = \omega_0^2 (2 - 2\cos qa)$

That's problem (a) solved, we've decoupled the oscillators.

Problem (b) is to define new operators

$$x_q = \left( \frac{\hbar}{2mc\omega_q} \right)^{\frac{1}{2}} (a_q + a_{-q}^\dagger) \quad p_q = -i \left( \frac{\hbar m \omega_q}{2} \right)^{\frac{1}{2}} (a_q - a_{-q}^\dagger)$$

Before we start, notice that

$$P_q^+ = P_{-q} \quad \text{and} \quad \omega_q^+ = \omega_{-q} \quad \text{for real } p_j \text{ and } \omega_j \quad [\text{prove this!}]$$

$$\begin{aligned} \hat{H} &= \sum_q \frac{1}{2m} \left( \frac{\hbar m \omega_q}{2} \right) (\hat{a}_q - \hat{a}_q^\dagger)(\hat{a}_q^\dagger - \hat{a}_q) + \frac{1}{2} m \omega_q^2 \left( \frac{\hbar}{2m \omega_q} \right) (\hat{a}_q + \hat{a}_{-q}^\dagger)(\hat{a}_q^\dagger + \hat{a}_{-q}) \\ &= \sum_q \frac{\hbar \omega_q}{4} \left[ \hat{a}_q \hat{a}_q^\dagger + \hat{a}_{-q}^\dagger \hat{a}_{-q} - \hat{a}_q^\dagger \hat{a}_q^\dagger - \hat{a}_q \hat{a}_{-q}^\dagger \right] \\ &\quad + \frac{\hbar \omega_q}{4} \left[ \hat{a}_q \hat{a}_q^\dagger + \hat{a}_q \hat{a}_{-q}^\dagger + \hat{a}_{-q}^\dagger \hat{a}_q^\dagger + \hat{a}_{-q}^\dagger \hat{a}_{-q} \right] \\ &= \sum_q \frac{\hbar \omega_q}{2} \left( \hat{a}_q \hat{a}_q^\dagger + \hat{a}_{-q}^\dagger \hat{a}_{-q} \right) \equiv \sum_q \frac{\hbar \omega_q}{2} \left( \hat{a}_q^\dagger \hat{a}_q + \hat{a}_q \hat{a}_q^\dagger \right) \quad [\text{reindexing the sum}] \end{aligned}$$

We set the commutation relations to be

$$[\hat{a}_q, \hat{a}_k^\dagger] = \delta_{qk}$$

so that operators for different wavevectors commute

We obtain

$$\hat{H} = \sum_q \frac{\hbar \omega_q}{2} (2\hat{a}_q^\dagger \hat{a}_q + 1) = \sum_q \hbar \omega_q \left( \hat{n}_q + \frac{1}{2} \right)$$

counts all quanta  
with wavevector  $q$

and we've finished successfully!

So far this has been a neat mathematical exercise, but it has a vivid interpretation which is the basis of the rest of the course

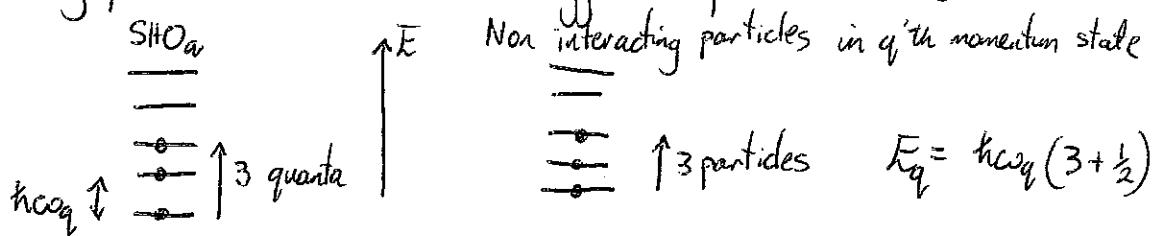
(Interacting particles)  $\longrightarrow$  (non interacting modes (waves))  $\xrightarrow{\text{Each mode is described by separate SHO creation}} \{\text{annihilation}\}$  operators

This motivates the particle interpretation

$$\left( \begin{array}{l} \text{no of quanta} \\ \text{in a SHO labelled } q \end{array} \right) \Rightarrow \left( \begin{array}{l} \text{no of non interacting} \\ \text{particles in a momentum} \\ \text{state } q \end{array} \right)$$

Q. Why does the particle interpretation work?

A. Non interacting particles have the same energy level spectrum as a SHO



$$\text{Total energy} = \sum_q h\omega_q (n_q + \frac{1}{2}) \quad \text{Applies to both cases}$$

↑ sum over all oscillators {momenta}

For the linked chain we call these 'quasiparticles' phonons.

- Recap
- \* Start with strongly interacting particles
  - \* Interpret the system as a set of non interacting quasiparticles
  - \* Describe quasiparticles with the machinery of  $a^\dagger$  and  $a$  operators

The procedure we followed went in 2 stages

$$(i) \text{Decouple into modes } p_j \rightarrow p_k \quad (ii) \text{Quantize modes } p_k \propto (a - a^\dagger)$$

Let's simplify the procedure by combining these steps

$$x_j = \frac{1}{\sqrt{N}} \sum_q \left( \frac{\hbar}{2m\omega_q} \right)^{\frac{1}{2}} (a_q + a_{-q}^\dagger) e^{iqja}$$

reindex the sum to obtain the 'mode expansion'

$$x_j = \left( \frac{\hbar}{mN} \right)^{\frac{1}{2}} \sum_q \left( \frac{1}{2\omega_q} \right)^{\frac{1}{2}} [a_q e^{iqja} + a_{-q}^\dagger e^{-iqja}]$$

↓                              ↑  
 annihilates                    creates  
 particles with                'antiparticles'  
 momenta  $h_q$                 with momenta  $-h_q$

- \* The mode expansion contains +ve (particle) and -ve (antiparticle) momentum parts
- \* For phonons particles & antiparticles are identical, but this isn't always the case
- \* In general, we define field operators which look like

$$x_j = \sum_q \left( \begin{array}{c} \text{annihilate} \\ \text{incoming} \\ \text{particle} \end{array} \right) + \left( \begin{array}{c} \text{create} \\ \text{outgoing} \\ \text{particle} \end{array} \right)_{-q}$$

## Lecture 2/3. Occupation numbers

We'll reformulate QM to be in line with the SHO formalism we used to solve the coupled oscillator problem.

We do 2 things (i) A (trivial) change of notation

(ii) A (more radical) change where we dispose of wavefunctions.

### 3.1] The trivial change in notation

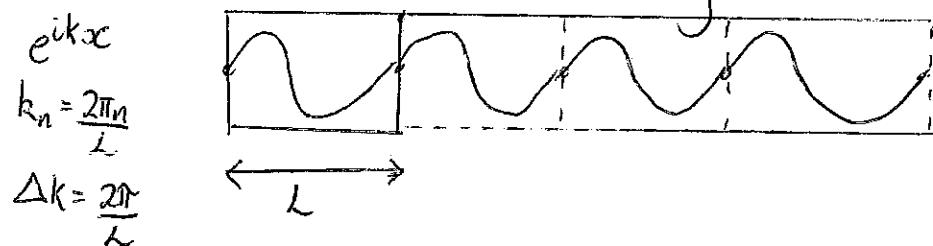
For the SHO we labelled states by the  $n^{\circ}$  of excitations in the oscillator  $|n\rangle$

If we have more oscillators (one on my desk, one in the park, one in the office next door...) we would write the states:

$$|N_{q_1} N_{q_2} N_{q_3} N_{q_4} \dots \rangle$$

$\begin{matrix} \nearrow & \nearrow & \nearrow & \nearrow \\ \text{oscillator } q_1 & \text{oscillator } q_2 & \text{oscillator } q_3 & \dots \end{matrix}$

Now consider a box with periodic boundary conditions



This applies to any system in a box  
[Electrons, phonons, photons ...]

Now make the particle interpretation

$$\left( \begin{array}{l} \text{independent} \\ \text{modes in a} \\ \text{box} \\ \text{momenta modes} \\ q_1, q_2, q_3 \dots \end{array} \right) \leftrightarrow \left( \begin{array}{l} \text{independent} \\ \text{oscillators} \\ \text{labelled} \\ q_1, q_2, q_3 \dots \end{array} \right)$$

$$\left( \begin{array}{l} \text{Identical particles} \\ \text{in a mode} \end{array} \right) \leftrightarrow \left( \begin{array}{l} \text{excitations in} \\ \text{an oscillator} \end{array} \right)$$

So we can write a state of identical particles as

$$|N_{q_1} N_{q_2} N_{q_3} \dots \rangle \quad \text{where } n_{q_i} \text{ is the } n^{\circ} \text{ of particles in momentum mode } q_i$$

To get the total energy (in either case) you can write

$$E = \sum_q \hbar \omega_q (n_q + \frac{1}{2})$$

↑      ↑      ↑      ↑      ↑

Total energy      energy level      spring of      number of excitations/particles in oscillator/mode q      mode/oscillator q

Remember: we are not saying that there's an oscillator at every point in the box. It's just an interpretation of the particles in modes as excitations of fictional oscillators. It works because we have

(i) Independent particles

(ii) Equally spaced oscillator/modes, energy levels

3.2 So much for the simple change. Now for the dramatic one - we're going to do away with wavefunctions all together.

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Why? Remember that for identical particles wavefunctions are at least as horrible as

$$\Psi(r_1, r_2) = \frac{1}{\sqrt{2}} [\phi_a(r_1)\phi_b(r_2) \pm \phi_b(r_1)\phi_a(r_2)]$$

(+ Bosons  
- Fermions)

\* We'll remove the action from the wavefunctions and just have the symmetry requirements imposed by  $a$  and  $a^+$  commutation relations.

Look to the SHO for inspiration. In that case we have

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle$$

vacuum

i.e. we could do away with all states except the vacuum

For oscillators plural, we build up states with

$$|n_{q_1}, n_{q_2} \dots\rangle = \prod_{n_q} \frac{(\alpha_q^+)^{n_q}}{\sqrt{n_q!}} |0\rangle$$

Example build the state  $|2_{q_1}, 1_{q_2}, 0_{q_3}\rangle = \left[ \frac{1}{\sqrt{2!}} (\alpha_{q_1}^+)^2 \right] \left[ \frac{1}{\sqrt{1!}} (\alpha_{q_2}^+) \right] |0\rangle$

Now we apply the philosophy to particles.

We'll need to ensure that we obey the QM rules for identical particles

① Bosons are symmetrical on exchange of particles

② Fermions are antisymmetrical on exchange

To implement these rules we need to worry about commutators & the order we put particles into states

Example  $\alpha_{p_2}^+ \alpha_{p_1}^+ |0\rangle \propto |l_{p_1} l_{p_2}\rangle \quad \alpha_{p_1}^+ \alpha_{p_2}^+ |0\rangle \propto |l_{p_2} l_{p_1}\rangle$

\*BUT\* the constants of proportionality will be different.

### 3.3 Bosons & Fermions

From the last example  $\alpha_1^+ \alpha_2^+ |0\rangle = \lambda \alpha_2^+ \alpha_1^+ |0\rangle$

The indistinguishability of particles constrains  $\lambda = \pm 1$  (why?)

We examine the two cases separately

$\lambda = 1$  Bosons

$$\alpha_1^+ \alpha_2^+ - \alpha_2^+ \alpha_1^+ |0\rangle = 0$$

implying  $[\alpha_1^+, \alpha_2^+] = 0$  (and also  $[\alpha_1, \alpha_2] = 0$ )

Operators creating/annihilating Bosons commute. This is just like the oscillator example (and photons are indeed Bosons!)

We also need  $[\alpha_1, \alpha_2^+] = \delta_{12}$  for the SHO analogy to work.

$\lambda = -1$  : Fermions

$$(\alpha_1^\dagger \alpha_2^\dagger + \alpha_2^\dagger \alpha_1^\dagger) |10\rangle = 0$$

Implying  $\{\alpha_1^\dagger, \alpha_2^\dagger\} = 0$  and  $\{\alpha_1, \alpha_2\} = 0$  Fermion operators anticommute  
 [This is an anticommutator  
 $\equiv \alpha_1^\dagger \alpha_2^\dagger + \alpha_2^\dagger \alpha_1^\dagger$ ]

In this case we'll need  $\{\alpha_1, \alpha_2^\dagger\} = \delta_{12}$  in order for things to work.

What is the consequence of anticommutation?

Try to put 2 particles in a state

$$\alpha_1^\dagger \alpha_1^\dagger |10\rangle = \underbrace{-\alpha_1^\dagger \alpha_1^\dagger}_{\text{Swapping operator order}} |10\rangle = 0$$

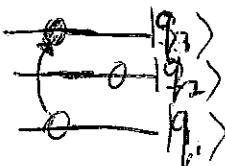
This is the Pauli principle - it's impossible to put 2 particles in a state

Some more examples

Let's try exchanging particles. We'll do this in stages

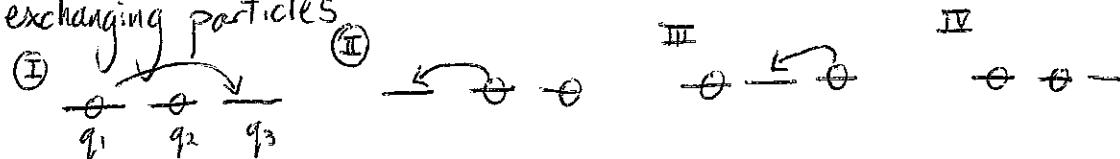


Each step involves killing a particle in one level & creating one in another.



Some more examples

Let's try exchanging particles



Each stage involves killing a particle in one ~~level~~<sup>mode</sup> & creating one in another!

$$\begin{array}{c} \text{---} \\ \theta \\ q_1 \end{array} \quad \begin{array}{c} \text{---} \\ \theta \\ q_2 \end{array} \quad \begin{array}{c} \text{---} \\ \theta \\ q_3 \end{array} = \begin{array}{c} \text{---} \\ \theta^+ \\ q_3 \end{array} \quad \begin{array}{c} \text{---} \\ \theta \\ q_1 \end{array} \quad \begin{array}{c} \text{---} \\ \theta \\ q_2 \end{array}$$

↑ kills particle in  $q_3$

creates particle in  $q_3$

The entire particle exchange process looks like

$$a_2^+ a_3^+ a_1^+ a_2 a_3^+ a_1 |1,1_2\rangle$$

We'll do this for Fermions, where swapping two operators of the same ~~sign~~<sup>generally</sup> gives a -ve sign  $\{a_i, a_j\} = 0$ ,  $\{a_i^+, a_j^+\} = 0$  but  $\{a_i, a_j^+\} = \delta_{ij}$

We get rid of operators by making number operators  $a_i^+ a_i$  which count the no. of particles in state  $i$

Exchanging fermions cont.

$$\begin{aligned}
 & a_2^+ a_3^+ a_1^+ a_2 a_3^+ a_1 |1,1_2\rangle \\
 &= -a_2^+ a_3^+ a_2 a_1^+ a_3^+ a_1 |1,1_2\rangle \quad \text{using } a_1^+ a_2 = -a_2 a_1^+ \\
 &= +a_2^+ a_3^+ a_2 a_3^+ (a_1^+ a_1) |1,1_2\rangle \quad \text{using } a_1^+ a_3^+ = -a_3^+ a_1^+ \\
 &= +a_2^+ a_3^+ a_2 a_3^+ |1,1_2\rangle \quad \text{since } a_1^+ a_1 |1,1_2\rangle = 1 \times |1,1_2\rangle \\
 &= -a_2^+ a_3^+ a_3^+ a_2 |1,1_2\rangle \\
 &= -a_3 a_3^+ a_2 a_2 |1,1_2\rangle \\
 &= -a_3 a_3^+ |1,1_2\rangle \\
 &= -(1 - a_3 a_3^+) |1,1_2\rangle \quad \text{since } a_3 a_3^+ + a_3^+ a_3 = 1 \\
 &= -|1,1_2\rangle \quad \text{since } a_3^+ a_3 |1,1_2\rangle = 0
 \end{aligned}$$

Which is correct. Swapping 2 particles gives us a minus sign.

Another example: recovering the caveman notation

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$$\Psi(x, y) = \langle xy | p_1 p_2 \rangle \quad \text{where } |xy\rangle = \frac{1}{\sqrt{2}} \sum_{q_1 q_2} \phi_{q_1}(x) \phi_{q_2}(y) |q_1 q_2\rangle$$

$$\text{So } \Psi(x, y) = \frac{1}{\sqrt{2}} \sum_{q_1 q_2} \phi_{q_1}(x) \phi_{q_2}(y) \langle q_1 q_2 | p_1 p_2 \rangle$$

So we need to find  $\langle q_1 q_2 | p_1 p_2 \rangle$ , which we'll do for Bosons

$$\begin{aligned} \langle q_1 q_2 | p_1 p_2 \rangle &= \langle 0 | a_{q_1} a_{q_2} a_{p_2}^\dagger a_{p_1}^\dagger | 0 \rangle \\ &= \langle 0 | a_{q_1} (a_{p_2}^\dagger a_{q_2}) a_{p_1}^\dagger | 0 \rangle + \delta(p_2 - q_2) \langle 0 | a_{q_1} a_{p_1}^\dagger | 0 \rangle \\ &= \langle 0 | a_{p_2}^\dagger a_{q_1} a_{q_2} a_{p_1}^\dagger | 0 \rangle + \delta(p_2 - q_2) \langle 0 | a_{q_2}^\dagger a_{p_1}^\dagger | 0 \rangle \\ &\quad + \delta(p_2 - q_2) \delta(q_1 - p_1) \\ &= \delta(p_2 - q_1) \delta(q_2 - p_1) + \delta(p_2 - q_2) \delta(q_1 - p_1) \end{aligned}$$

This is zero  
as it contains  
 $\langle 0 | a^\dagger = 0$

Insert  $\langle q_1 q_2 | p_1 p_2 \rangle$  into the expansion

$$\begin{aligned} \Psi(x, y) &= \frac{1}{\sqrt{2}} \sum_{q_1 q_2} \phi_{q_1}(x) \phi_{q_2}(y) [\delta(p_1 - q_1) \delta(p_2 - q_2) + \delta(p_1 - q_2) \delta(p_2 - q_1)] \\ &= \frac{1}{\sqrt{2}} [\phi_{p_1}(x) \phi_{p_2}(y) + \phi_{p_2}(x) \phi_{p_1}(y)] \end{aligned}$$

So it works!

Summary

- \* The particle/mode interpretation used occupation numbers  $|n_{q_1} n_{q_2} n_{q_3} \dots\rangle$

- \* We don't use any states except  $|0\rangle$  and act on it with  $a_q$  and  $a_q^\dagger$

- \* Boson operators commute. \* Fermion operators anticommute

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