

Part 3 Some new machinery: the time evolution operator

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We do quantum mechanics by thinking about amplitudes. These are read from right-to-left (like Hebrew).

* The amplitude for ~~is~~ having a state $|\circlearrowleft\rangle$ at a time $t=t_{\text{smiley}}$ is

$$\langle \circlearrowleft | \psi(t_{\text{smiley}}) \rangle$$

* The expectation value of an operator $\hat{\mathcal{O}}$ is $\langle \psi | \hat{\mathcal{O}} | \psi \rangle$

However we do our Q.M. we should agree on these quantities.

Schrodinger's Q.M. relies on a function $\psi(t)$ which evolves in time according to $i \frac{\partial \psi}{\partial t} = \hat{H} \psi$.

If we treat \hat{H} like a number then we can integrate and obtain

$$\psi(t_2) = e^{-i\hat{H}(t_2-t_1)} \psi(t_1)$$

we call $\hat{U}(t_2, t_1) = e^{-i\hat{H}(t_2-t_1)}$ the time-evolution operator.

Some properties of $\hat{U}(t_2, t_1)$

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I To preserve probabilities \hat{U} is unitary

$$U^{-1} = U^+$$

II It can do nothing

$$\hat{U}(t_1, t_1) = 1$$

III It has a composition law $\hat{U}(t_3, t_2) \hat{U}(t_2, t_1) = \hat{U}(t_3, t_1)$

IV It has an inverse $\hat{U}(t_2, t_1)^{-1} = \hat{U}(t_1, t_2)$ [$= U^+(t_2, t_1)$ from I]

We can use \hat{U} to reformulate Q.M.

[Aside: you can also show $i \frac{\partial \hat{U}}{\partial t} = \hat{H} \hat{U}$, that is it obeys Schrodinger's eqn]

Reformulating quantum mechanics

In Schrodinger's picture of the world

- Schrodinger has [time dependent wavefunctions
time independent]

$$\langle O \rangle = \langle \psi(t) | \hat{O} | \psi(t) \rangle$$

$$[\text{where } \psi(t) = e^{-i\hat{H}t} \psi(0)] \\ \uparrow U(t, 0)$$

so we can write $\langle O \rangle = \langle \psi(0) e^{i\hat{H}t} \hat{O} e^{-i\hat{H}t} \psi(0) \rangle$

What if we move the braces?

$$\langle O \rangle = \langle \psi(0) | e^{i\hat{H}t} \hat{O} e^{-i\hat{H}t} | \psi(0) \rangle$$

- Now we have [time independent wavefunctions $\psi(0)$]

time dependent operators. $\hat{O}_H(t) = e^{i\hat{H}t} \hat{O} e^{-i\hat{H}t}$

This is known as the 'Heisenberg picture' of Q.M. It's a lot more like classical mechanics where dynamic variables like $x(t), p(t)$ are time dependent

Q: How do we get the eqn's of motion for the Heisenberg operators?

$$A: \frac{d \hat{O}_H(t)}{dt} = -i [\hat{O}_H(t), \hat{H}]$$

Recap: there's more than 1 way of thinking about Q.M

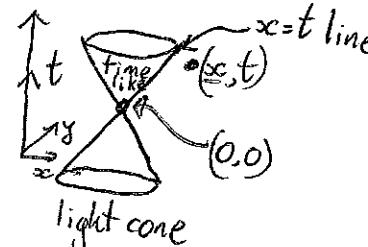
- In Schrodinger's picture wavefunctions are time dependent ~~$\psi(t) = \psi(0) e^{-i\hat{H}t}$~~
- Wavefunctions time evolve according to the Schrodinger eqn.
- In Heisenberg's picture operators are time dependent $\hat{O}_H(t) = e^{i\hat{H}t} \hat{O} e^{-i\hat{H}t}$
- Heisenberg's operators time evolve according to

$$\frac{d \hat{O}_H(t)}{dt} = -i [\hat{O}_H(t), \hat{H}]$$

We're going to use the time evolution operator to kill off non-relativistic QM!

We'll find the amplitude

$$A = \langle \underset{\text{end up at } \underline{x}[\text{at time } t]}{\underline{x}} | \hat{U}(t, 0) | \underset{\text{start at } \underline{x}=0 \text{ [at time } t=0]}{\underline{x}} \rangle$$

What if $\underline{x}_f = (\underline{x}, t)$ is spacelike separated from $(0, 0)$? 

We need A to be zero for spacelike separations
If not then we can observe a particle outside its forward light cone and QM is DEAD!

Let's see. We'll find

$$A = \langle \underline{x} | e^{-i\hat{H}t} | 0 \rangle \quad \text{with } \hat{H}|p\rangle = E_p |p\rangle$$

$$\text{and we'll need } \langle \underline{x}|p\rangle = \frac{1}{(2\pi)^{3/2}} e^{ip \cdot \underline{x}}$$

$$\begin{aligned} A &= \langle \underline{x} | e^{-i\hat{H}t} | 0 \rangle \\ &= \int d^3 p \langle \underline{x} | e^{-i\hat{H}t} | p \rangle \langle p | 0 \rangle \\ &= \int d^3 p \langle \underline{x} | e^{-iE_p t} | p \rangle \frac{e^{-ip \cdot (\underline{x}=0)}}{(2\pi)^{3/2}} \\ &= \int d^3 p \langle \underline{x}|p\rangle e^{-iE_p t} \frac{1}{(2\pi)^{3/2}} \\ &= \int d^3 p \frac{e^{ip \cdot \underline{x}}}{(2\pi)^3} e^{-iE_p t} = \int \frac{d^3 p}{(2\pi)^3} e^{i[p \cdot \underline{x} - (p^2 + m^2)^{1/2} t]} \end{aligned}$$

This integral can be done [see handout]

The answer is

$$A = \frac{i}{2\pi^2 z^2} e^{-m|z|} \int_m^\infty dz e^{-(z-m)^2 t} \sinh(z^2 - m^2)^{\frac{1}{2}} t$$

+ve definite

So $A \sim e^{-m|z|} > 0$ and there is a nonzero probability of observing a particle outside its lightcone. QM is DEAD!

Q. What are we going to do?

A. We need fields.

- * This terrible tragedy occurred because we assumed that we / an observer could see everything. This isn't true.
- * Observers can only make observations locally. Operators need to be functions of position in space-time $\hat{\mathcal{O}}(x)$.
- * This is the same as Classical Field theories such as electromagnetism. In that theory we have $E(x) \propto B(x)$, which tell us about fields at some point x in spacetime.
- * We're going to define operator valued fields $\hat{\mathcal{O}}(x)$ which allow us to make measurements at a point x .
- * To ensure that no information travels at speeds $> c$ we must have

$$[\hat{\mathcal{O}}(x_1), \hat{\mathcal{O}}(x_2)] = 0 \quad \text{for } |x_1 - x_2|^2 < 0$$

In that case measurements separated by space-like intervals won't influence each other

We'll now find these quantum fields...

Supplement

Doing the integral

$$A = \int \frac{d^3 p}{(2\pi)^3} e^{i[p \cdot \vec{x} - (p^2 + m^2)t]}$$

Stage I: convert to spherical polars

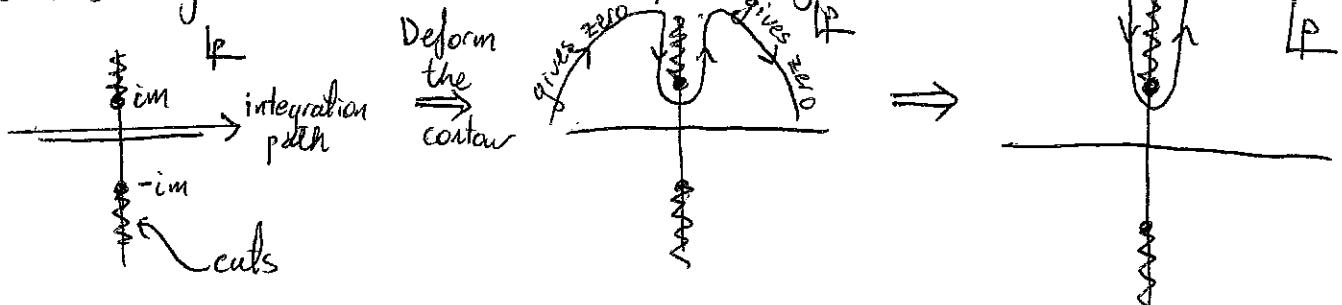
$$A = \int_0^{2\pi} d\phi \int_0^\infty \frac{dp}{(2\pi)^3} \int_{-1}^1 d(\cos\theta) p^2 e^{ip|\vec{x}| \cos\theta} e^{-iE_p t}$$

$$\text{with } E_p = (p^2 + m^2)^{\frac{1}{2}}$$

$$A = \frac{1}{(2\pi)^2} \frac{1}{i|p|c|} \int_0^\infty dp p \left(e^{ip|\vec{x}|} - e^{-ip|\vec{x}|} \right) e^{-iE_p t}$$

$$= \frac{-i}{(2\pi)^2 |p|c} \int_{-\infty}^\infty pdp e^{ip|\vec{x}|} e^{-iE_p t} = \frac{-i}{(2\pi)^2 |p|c} \int_{-\infty}^\infty pdp e^{ip|\vec{x}|} e^{-i(p^2 + m^2)^{\frac{1}{2}}t}$$

To do this integral we'll need some complex analysis



The hemisphere gives zero because (i) $|x| > 0$: $e^{ip|\vec{x}|}$ decreases as the radius increases

Lastly, substitute $p = iz$ to get

- (ii) $e^{-i(p^2 + m^2)^{\frac{1}{2}}t}$ decreases on the left side
- (iii) $e^{-i(p^2 + m^2)^{\frac{1}{2}}t}$ increases on the right, but since $|x| > t$ the product of exponentials decreases

$$A = \frac{-i}{(2\pi)^2 |p|c} \int_m^\infty iz dz (iz) e^{z|\vec{x}|} \left[e^{(z^2 - m^2)^{\frac{1}{2}}t} - e^{-(z^2 - m^2)^{\frac{1}{2}}t} \right]$$

$$= \frac{i}{(2\pi)^2 |p|c} e^{-mp|\vec{x}|} \int_m^\infty dz e^{-(z-m)|\vec{x}|} 2 \sinh((z^2 - m^2)^{\frac{1}{2}}t)$$

as we had in
the notes.

Part 4 Classical fields + classical particles

We need to know about these to be able to deal with the quantum versions.

Recap of classical particle mechanics

We describe classical particles with Lagrangians

$$L = \begin{pmatrix} \text{kinetic} \\ \text{energy} \end{pmatrix} - \begin{pmatrix} \text{Potential} \\ \text{energy} \end{pmatrix}$$

L is a function of positions q_i , velocities \dot{q}_i (and possibly time t)

The Lagrangian is useful because of Hamilton's principle which says that

$S = \int_{t_a}^{t_b} L dt$ is a minimum for a trajectory taking place between t_a and t_b .

$$\text{i.e. } SS = 0 = \int_{t_a}^{t_b} \delta L dt = \int_{t_a}^{t_b} dt \left[\left(\frac{\partial L}{\partial q} \right) \delta q + \left(\frac{\partial L}{\partial \dot{q}} \right) \delta \dot{q} \right]$$

$$= \int_{t_a}^{t_b} dt \left(\frac{\partial L}{\partial q} \right) \delta q + \left(\frac{\partial L}{\partial \dot{q}} \right) \frac{d}{dt} (\delta q)$$

Integrate the second term by parts $u = \frac{\partial L}{\partial \dot{q}}$ $v = \frac{d}{dt} (\delta q)$

$$= \int_{t_a}^{t_b} dt \left(\frac{\partial L}{\partial q} \right) \delta q + \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_a}^{t_b} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q$$

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but we set $\delta q = 0$ at t_a & t_b so the 2nd term is zero & we obtain

$$SS = 0 = \int_{t_a}^{t_b} \delta q \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right] dt$$

Implyind

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0}$$

This is the Euler-Lagrange equation

In addition, we define the momentum

$$p = \left(\frac{\partial L}{\partial \dot{q}} \right)$$

and the Hamiltonian $H = p\dot{q} - L$

Example: The SHO has $L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega_0^2 q^2$

$$\frac{\partial L}{\partial q} = -m\omega_0^2 q \quad \frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = m\ddot{q}$$

Plugging in to the E-L equation yields the equation of motion

$$\begin{aligned} \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} &= 0 \\ -m\omega_0^2 q - m\ddot{q} &= 0 \\ \ddot{q} &= -\omega_0^2 q \quad \text{as usual} \end{aligned}$$

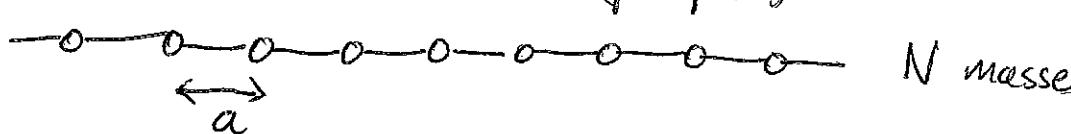
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$$\begin{aligned} H &= p\dot{q} - L \\ &= \left(\frac{\partial L}{\partial \dot{q}} \right) \dot{q} - L = m(\dot{q})^2 - \frac{1}{2}m(\dot{q})^2 + \frac{1}{2}m\omega_0^2 q^2 \\ &= \frac{1}{2}m(\dot{q})^2 + \frac{1}{2}m\omega_0^2 q^2 \quad \text{which is correct.} \end{aligned}$$

Next: Classical fields

We can apply this to a chain of Harmonic oscillators.

It is easy to write down

$$L = \sum_i^N \frac{1}{2}m(\dot{q}_i)^2 - \frac{1}{2}m\omega_0^2 (q_i - q_{i+1})^2$$


N masses

What happens in the continuum limit, i.e. as $a \rightarrow 0 \times N \rightarrow \infty$?

Rules
for the
continuum
limit

$$\sum_i \rightarrow \frac{1}{a} \int dx$$

$$q_i \rightarrow \phi(x)$$

$$\frac{(q_i - q_{i-1})}{a} \rightarrow \frac{\partial \phi(x)}{\partial x}$$

This is a field! It's a machine
that takes an input position x
and outputs an amplitude $\phi(x)$

we
obtain

$$L = \int \frac{dx}{a} \left\{ \frac{m}{2} \left[\frac{d}{dt} \phi(x) \right]^2 - \frac{mc\omega_0^2}{2} \left(\frac{\partial \phi(x)}{\partial x} \right)^2 a^2 \right\}$$

Define $\rho = \frac{m}{a} = \frac{\text{mass}}{\text{density}}$ $\tau = mc\omega_0^2 a^2 = \text{tension}$ and set $e = v = 1$
for maximum simplicity

$$L = \int dx \left[\frac{\rho}{2} \left(\frac{\partial \phi(x)}{\partial t} \right)^2 - \frac{\tau}{2} \left(\frac{\partial \phi(x)}{\partial x} \right)^2 \right]$$

$$= \int dx \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 \right]$$

Define a 'Lagrangian density' such that $L = \int dx L[\phi(x)]$

$L[\phi(x)]$ is a 'functional'. It ~~takes a function $\phi(x)$ and outputs a number~~. We can use 4d notation to simplify things further.

$$L = \int dx \frac{1}{2} (\partial^\mu \phi)(\partial_\mu \phi) = \int dx \frac{1}{2} (\partial_\mu \phi)^2$$

[See the notation guide if you're not sure about this]

Just as we had the E-L equations for L , we also have them for \mathcal{L} .

$$\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} = 0$$

These are functional derivatives [see supplement]

Try these out on the Lagrange density for the string

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi)$$

$$\frac{\delta \mathcal{L}}{\delta \phi} = 0 \quad \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} = (\partial^\mu \phi)$$

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi \\ = \partial^2 \phi$$

So the eqn of motion is $\partial^2 \phi = 0$

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0 \quad \text{i.e. the wave eqn.}$$

The general solutions are given by

$$\phi = \int \frac{dk}{(2\pi)^{1/2}} \frac{1}{(2\omega_k)^{1/2}} [a_k e^{-i\omega_k t + ik \cdot x} + a_k^+ e^{i\omega_k t - ik \cdot x}]$$

We also have a momentum density $\Pi^0 = \frac{\delta \mathcal{L}}{\delta (\partial_0 \phi)} = \partial^0 \phi = \frac{\partial \phi}{\partial t}$

and a Hamiltonian density $H = \Pi^0 (\partial_0 \phi) - \mathcal{L}$

$$\begin{aligned} &= \partial^0 \phi \partial_0 \phi - \frac{(\partial_\mu \phi)^2}{2} \\ &= \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 \end{aligned}$$

Time for a second example : the 2nd simplest field theory

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2$$

\uparrow
Lagrangian
from the string
example

\star
a 'potential
energy' term

Let's try out the E-L equations again...

$$\frac{\delta \mathcal{L}}{\delta \phi} = -m^2 \phi$$

$$\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} = \partial^\mu \phi$$

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi = \partial^2 \phi$$

so we obtain $\partial^2 \phi + m^2 \phi = 0$

$$(\partial^2 + m^2) \phi = 0 \text{ The Klein-Gordon equation!}$$

Amazingly the K.G. equation pops out of the simplest field theory we can write down.

Where next?

We're going to quantize these classical fields. We'll expand the fields in terms of quantum mechanical modes & then make a practical interpretation.

A brief tour of field theories

(I) $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2$ The wave equation. Particles are massless $E_p = c_p$
 $\partial^2 \phi = 0$

(II) $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2$ The scalar field Lagrangian. Particles are scalar mesons $E_p = p^2 + m^2$
 $(\partial^2 + m^2) \phi = 0$

(III) $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \mu^2 \phi \phi^+$ complex scalar field theory

(IV) $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{1}{4!} \phi^4$ Phi-four theory. Used to model phase transitions. Can't be solved exactly

(V) $\psi^+ \psi \phi$ theory:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 + (\partial_\mu \psi)(\partial^\mu \psi^+) - \mu^2 \psi \psi^+ - g \psi \psi^+ \phi$$

is a simplified version of Q.E.D.

There are loads more!