

Part 5 Canonical Quantization

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A method to turn a classical field theory into a quantum field theory
It's a simple 'turn-the-crank' method of 5 stages

① Write down $\mathcal{L}[\phi]$

② Calculate momentum density $\Pi^0[\phi]$ and Hamiltonian H

③ Impose commutation relations $[\hat{\phi}(x), \hat{\Pi}^0(y)] = i\delta^{(3)}(x-y)$

④ Expand fields in a mode expansion where amplitudes are \hat{a}^\pm

⑤ That's it, as long as you remember the normal ordering interpretation

We'll use the 5-point plan to quantize scalar field theory. This will work exactly because the Lagrangian is (bi)quadratic in the field & its derivatives. In more complicated cases we'll need a perturbation theory.

Step I $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2$

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Step II The momentum is $\Pi^0 = \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi)}$. Easiest just to take the 0^{th}

component of $\Pi^\mu = \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} = \partial^\mu \phi$. So $\boxed{\Pi^0 = \partial^0 \phi}$

The Hamiltonian is given by $H = \Pi^0 \partial_0 \phi - \mathcal{L}$

$$\begin{aligned} &= \partial^0 \phi \partial_0 \phi - \frac{1}{2}(\partial_0 \phi \partial^0 \phi + \frac{1}{2} \partial_i \phi \partial^i \phi + \frac{m^2}{2} \phi^2) \\ &= \frac{1}{2} \partial^0 \phi \partial_0 \phi + \frac{1}{2} \partial_i \phi \partial^i \phi + \frac{m^2}{2} \phi^2 \quad (i=1,2,3) \\ &= (\text{kinetic energy}) + (\text{potential energy})^2 \end{aligned}$$

[Note that $\partial_i = \nabla$, $\partial^i = -\nabla$]

Step III Make these classical fields quantum

$$[\hat{\phi}(x), \hat{\Pi}^0(y)] = i\delta^{(3)}(x-y)$$

At equal times only

Step IV Make the mode expansion

Remember how the phonon problem included the expansion

$$\hat{\xi}_j = \sum_q \left(\frac{\hbar}{m}\right)^{1/2} \frac{1}{(2\omega_q)^{1/2}} [\hat{a}_q e^{iq_0 x} + \hat{a}_q^\dagger e^{-iq_0 x}]$$

We'll do the same thing here. The answer will be

$$\hat{\Phi}(x) = \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2E_p)} [\hat{a}_p e^{ip_0 x} + \hat{a}_p^\dagger e^{-ip_0 x}]$$

↑ Integration measure ↑ normalization factor: $E_p^2 = p^2 + m^2$ ↑ annihilates particles ↑ creates antiparticles

This is quite significant, so we'll pause to take a closer look at the meaning of $\hat{\Phi}(x)$

$\hat{\Phi}(x)$ is an operator which acts on the vacuum $|0\rangle$ creating antiparticles and annihilating particles

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In single particle Q.M

$$|\psi\rangle = \sum_p \langle p | \psi \rangle |p\rangle$$

↓ (state) ↑ (amplitude) ↓ (basis state)

$$\text{In position space } \langle x | \psi \rangle = \sum_p \langle p | \psi \rangle \langle x | p \rangle$$

$$\Psi(x) = \sum_p a_p e^{ip_0 x}$$

this is the wavefunction

Second quantization corresponds to turning wavefunctions into operators

$$\hat{\Psi}(x) = \sum_p \hat{a}_p e^{-ip_0 x} \rightarrow \text{the amplitude becomes an annihilation operator, with } [\hat{a}_p, \hat{a}_{p'}^\dagger] = \delta^{(3)}(p - p')$$

$\hat{\Psi}(x)$ annihilates a superposition of momentum eigenstate particles at the position x . \hat{a}_p is the annihilation operator that does the murdering

Similarly $\hat{\psi}^+(\underline{x}) = \sum_f \hat{a}_f^+ e^{ip \cdot \underline{x}}$ creates a particle at position \underline{x} .

Q: Why do we need a particle & antiparticle part?

A: This deals with the ~~extra~~ negative energy problem

So the answer is

$$\hat{\phi}(\underline{x}) = \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2E_p)^{1/2}} [\hat{a}_p e^{ip \cdot \underline{x}} + \hat{a}_p^\dagger e^{-ip \cdot \underline{x}}]$$

We're going to work in the Heisenberg picture, so we need to make this operator time dependent

$$\hat{\phi}(\underline{x}) = \hat{\phi}(t, \underline{x}) = e^{i\hat{H}t} \hat{\phi}(\underline{x}) e^{-i\hat{H}t}$$

gives $\hat{\phi}(\underline{x}) = \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2E_p)^{1/2}} [\hat{a}_p e^{-ip_0 \cdot \underline{x}} + \hat{a}_p^\dagger e^{ip_0 \cdot \underline{x}}]$

where $p_0 \cdot \underline{x} = Et - \underline{p} \cdot \underline{x}$ ie the 4d dot product

$[\hat{a}_p, \hat{a}_q^\dagger] = \delta(p-q)$ at equal times and $E_p = (\underline{p}^2 + m^2)^{1/2}$ (tve roots only)

Q: What's the relationship between $\hat{\phi}(\underline{x})$ & the wavefunction from single particle QM?

A: Act on $|0\rangle$ with $\hat{\phi}^+(\underline{x})$ creates superpositions

$$\hat{\phi}^+(\underline{x}) |0\rangle = \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2E_p)^{1/2}} \hat{a}_p^\dagger |0\rangle e^{ip \cdot \underline{x}}$$

Fold in a $\langle q|$ & obtain: $\langle q | \hat{\phi}^+(\underline{x}) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2E_p)^{1/2}} \langle q | p \rangle e^{ip \cdot \underline{x}}$

$$= \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2E_p)^{1/2}} \delta(p-q) e^{ip \cdot \underline{x}}$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{1}{(2E_q)^{1/2}} e^{iq \cdot \underline{x}}$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{1}{(2E_q)^{1/2}} e^{i[E_q t - q \cdot \underline{x}]}$$

= a single particle wavefunction for a particle in the q th momentum mode
normalized so that there are

Step IV continued : inserting the mode expansion

$$\text{Insert } \hat{\phi}(x) = \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2\varepsilon_p)^{1/2}} [\hat{a}_p e^{-ip \cdot x} + \hat{a}_p^\dagger e^{ip \cdot x}]$$

$$\text{into } \hat{H} = \frac{1}{2} (\partial^0 \hat{\phi}) (\partial_0 \hat{\phi}) + \frac{1}{2} (\partial_i \hat{\phi}) (\partial^i \hat{\phi}) + \frac{m^2}{2} \hat{\phi}^2$$

Nothing for it but to plug away

(i) Momentum $\hat{P}^0 = \partial^0 \hat{\phi} = \partial_0 \hat{\phi}$ [no difference between up & down time-like indices]

$$\begin{aligned} &= \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2\varepsilon_p)^{1/2}} [-i\varepsilon_p \hat{a}_p e^{-ip \cdot x} + i\varepsilon_p \hat{a}_p^\dagger e^{ip \cdot x}] \\ &= \int \frac{d^3 p}{(2\pi)^{3/2}} -i\left(\frac{\varepsilon_p}{2}\right)^{1/2} [\hat{a}_p e^{-ip \cdot x} - \hat{a}_p^\dagger e^{ip \cdot x}] \end{aligned}$$

$$\begin{aligned} \text{(ii) Gradient } \partial_i \hat{\phi} &= \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2\varepsilon_p)^{1/2}} [ip \hat{a}_p e^{-ip \cdot x} - ip \hat{a}_p^\dagger e^{ip \cdot x}] \\ &= \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{i p}{(2\varepsilon_p)^{1/2}} [\hat{a}_p e^{-ip \cdot x} - \hat{a}_p^\dagger e^{ip \cdot x}] \end{aligned}$$

(iii) Next we'll look at the 4 terms thrown up by $[\hat{\phi}(x)]^2$

$$[\hat{\phi}(x)]^2 = \hat{a}_p \hat{a}_q e^{-i(p+q) \cdot x} + \hat{a}_p^\dagger \hat{a}_q^\dagger e^{i(p+q) \cdot x} + \hat{a}_p \hat{a}_q^\dagger e^{-i(p-q) \cdot x} + \hat{a}_p^\dagger \hat{a}_q e^{i(p-q) \cdot x}$$

We're after the Hamiltonian proper $H = \int \mathcal{H} d^3 x$, so we'll have 3 integrals to do.

$$\iiint \frac{d^3 p}{(2\pi)^{3/2}} \frac{d^3 q}{(2\pi)^{3/2}} \frac{d^3 x}{(2\varepsilon_p 2\varepsilon_q)^{1/2}} \times (4 \text{ terms})$$

As in the phonon problem we do this in 2 stages (a) Do the space integral (b) Do one of the momentum integrals

$$(a) : \int d^3 x [\phi(x)]^2 = \int \frac{d^3 p d^3 q}{(2\pi)^3 (2\varepsilon_p 2\varepsilon_q)^{1/2}} \times$$

$$\hat{a}_p \hat{a}_q \delta^{(3)}(p+q)(2\pi)^3$$

$$\hat{a}_p^\dagger \hat{a}_q^\dagger \delta^{(3)}(p+q)(2\pi)^3$$

$$\hat{a}_p \hat{a}_q^\dagger \delta^{(3)}(p-q)(2\pi)^3$$

$$\hat{a}_p^\dagger \hat{a}_q \delta^{(3)}(p-q)(2\pi)^3$$

This terms will vanish later in the calculation
We'll drop them here

(b) Use the δ functions to do one of the momentum integrals.

Repeat for the other terms to obtain

$$\begin{aligned} H &= \frac{1}{2} \int \frac{d^3 p}{2\epsilon_p} [\epsilon_p^2 + p^2 + m^2] (\hat{a}_p \hat{a}_p^\dagger + \hat{a}_p^\dagger \hat{a}_p) \\ &= \frac{1}{2} \int d^3 p \epsilon_p [\hat{a}_p \hat{a}_p^\dagger + \hat{a}_p^\dagger \hat{a}_p] \\ &= \cancel{\int d^3 p \epsilon_p [\hat{a}_p^\dagger \hat{a}_p + \frac{1}{2} \delta(0)]} \end{aligned}$$

The δ -function should fill us with fear - it yields ∞ . This is disgusting - but not a disaster. After all, we can't measure absolute energy-only energy differences [like in electrostatics where we measure p.d.'s]

In order to remove these meaningless ∞ 's we need a procedure that makes sense of strings of operators...

It's called NORMAL ORDERING

$$N[AB^\dagger CD^\dagger] = \left(\begin{array}{l} \text{operators arranged with all} \\ \text{creation operators on the left} \\ \text{[keep -signs, discard } \delta \text{ functions]} \end{array} \right)$$

We obtain

$$N[\hat{H}] = \int d^3 p \epsilon_p \hat{a}_p^\dagger \hat{a}_p \quad \begin{array}{l} \text{i.e. equally spaced} \\ \text{SHO-like energy levels.} \end{array}$$

We've turned the Lagrangian of non interacting particles of mass m with dispersion $\epsilon_p = \sqrt{p^2 + m^2}$

Summary: We have a 5 stage method that outputs a QFT

- (I) Write $L[\phi]$; (II) Calculate $\Pi^\circ[\phi]$ and $H[\phi]$
- (III) Impose commutation relations $[\phi, \Pi] = i\delta$; (IV) Expand in SHO-like modes
- (V) Normal order.

This solves theories quadratic in ϕ and $(\partial\phi)$. Other theories need perturbation theory.

Part 6 The Joy of 2nd Quantization

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In many problems we start with a '1st Quantized' Hamiltonian. We'd like to apply our field operators to these problems.

* Let's deal with nonrelativistic fermions in a box (\equiv a metal, of sorts)

The mode expansion is $\hat{\Psi}(\vec{x}) = \sum_p e^{ip\cdot\vec{x}}$

$$\text{with dispersion } E_p = \frac{p^2}{2m}$$

We could start by dreaming up a Lagrangian such as

$$\mathcal{L} = i\hbar \bar{\psi}^\dagger \gamma^\mu \psi - \frac{\hbar^2}{2m} \nabla \bar{\psi}^\dagger \cdot \nabla \psi - V(x) (\bar{\psi}^\dagger \psi)^2 \text{ and use the}$$

to canonical quantization method. This will result in the correct answer but there is a more direct way

Take the Hamiltonian to be

$$\hat{H} = \int d^3x \hat{\Psi}^\dagger(\vec{x}) \hat{H}_0 \hat{\Psi}(\vec{x})$$

↙
 2nd quantized ↘
 old 1st quantized
 single particle

field operators

$$\text{Example : the free particle } \hat{H}_0 = \frac{\hat{p}^2}{2m} = -\frac{\nabla^2}{2m}$$

$$\hat{H} = \frac{1}{N} \int d^3x \sum_{pq} \hat{a}_q^\dagger e^{-iq\cdot\vec{x}} \left(-\frac{\nabla^2}{2m} \right) \hat{a}_p e^{ip\cdot\vec{x}}$$

$$= \frac{1}{N} \int d^3x \sum_{pq} \hat{a}_q^\dagger \hat{a}_p \left(\frac{p^2}{2m} \right) e^{-i(q-p)\cdot\vec{x}} \quad [\text{Now do the space integral}]$$

$$= \frac{1}{N} \cancel{\int d^3x} \sum_{pq} \hat{a}_q^\dagger \hat{a}_p \frac{p^2}{2m} N \delta^{(3)}(q-p) \quad [\text{Now do one of the momentum sums}]$$

$$= \sum_p \frac{p^2}{2m} \hat{a}_p^\dagger \hat{a}_p$$

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Giving $\hat{H} = \sum_f \hat{a}_f^\dagger \hat{a}_f \frac{p^2}{2m} = \sum_f \hat{n}_f E_f$ energy of state f

\uparrow
no of particles
in state f

Example 2 An external potential

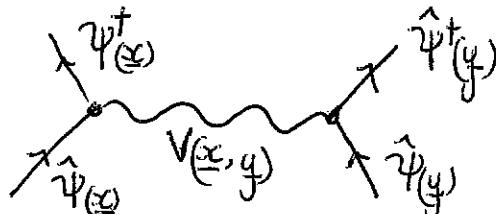
$$\begin{aligned}\hat{V} &= \int d^3x \hat{\Psi}^\dagger(\underline{x}) V(\underline{x}) \hat{\Psi}(\underline{x}) \quad \text{or in pictures} \quad \begin{array}{c} \uparrow \hat{\Psi}^\dagger(\underline{x}) \\ \downarrow \hat{\Psi}(\underline{x}) \end{array} \quad \begin{array}{l} \text{particle out} \\ \uparrow \\ \text{Interaction} \\ \downarrow \\ \text{particle in} \end{array} \\ &= \frac{1}{N} \int d^3x \sum_{fq} \hat{a}_q^\dagger e^{-iq \cdot \underline{x}} V(\underline{x}) \hat{a}_f e^{ip \cdot \underline{x}} \\ &= \frac{1}{N} \sum_{fq} \hat{a}_q^\dagger \hat{a}_f \left[\frac{1}{N} \int d^3x e^{-iq \cdot \underline{x}} V(\underline{x}) e^{ip \cdot \underline{x}} \right] = \sum_{fq} \hat{a}_q^\dagger \hat{a}_f \frac{1}{N} \int d^3x V(\underline{x}) x e^{-i(q-p) \cdot \underline{x}} \\ &= \sum_{fq} \hat{a}_q^\dagger \hat{a}_f \tilde{V}(q-f) \quad \text{or in pictures} \quad \begin{array}{c} \uparrow q \text{ particle leaves} \\ \downarrow f \text{ particle enters} \end{array} \quad \begin{array}{c} \text{potential acts} \\ \uparrow \end{array}\end{aligned}$$

$\text{In general } \hat{A}^{(1)} = \frac{1}{N} \int d^3x \sum_{fq} \hat{a}_q^\dagger e^{-iq \cdot \underline{x}} A(\underline{x}) \hat{a}_f e^{ip \cdot \underline{x}}$

So much for single particles, what about 2 particles interacting?
 This is really important - nearly all CMP is based on this idea

$$\hat{V} = \frac{1}{2} \int d^3x d^3y \hat{\Psi}^\dagger(\underline{x}) \hat{\Psi}^\dagger(\underline{y}) V(\underline{x}, \underline{y}) \hat{\Psi}(\underline{y}) \hat{\Psi}(\underline{x})$$

In pictures



Remember the order (like a dance step): left in, right in, right out, left out.

We'll insert of mode expansion to get a momentum space version

$$\hat{V} = \frac{1}{2N^2} \int d^3x d^3y \sum_{ff'q'q} e^{-iq \cdot \underline{x}} e^{-if \cdot \underline{y}} e^{iq' \cdot \underline{y}} e^{ip \cdot \underline{x}} \hat{a}_q^\dagger \hat{a}_f^\dagger \hat{a}_{q'} \hat{a}_f V(\underline{x} - \underline{y})$$

We assume space is isotropic so we can replace $V(\underline{x} - \underline{y}) \rightarrow V(\underline{x} - \underline{y})$

Let $\underline{z} + \underline{y} = \underline{x}$ and eliminate \underline{x}

$$\hat{V} = \frac{1}{2N^2} \int d^3 z d^3 y \sum_{\underline{p}, \underline{q}, \underline{f}, \underline{g}} e^{-i(\underline{g}-\underline{p}) \cdot \underline{z}} e^{-i(\underline{g}+\underline{f}-\underline{q}-\underline{p}) \cdot \underline{y}} V(\underline{z}) a_g^+ a_f^+ a_q a_p$$

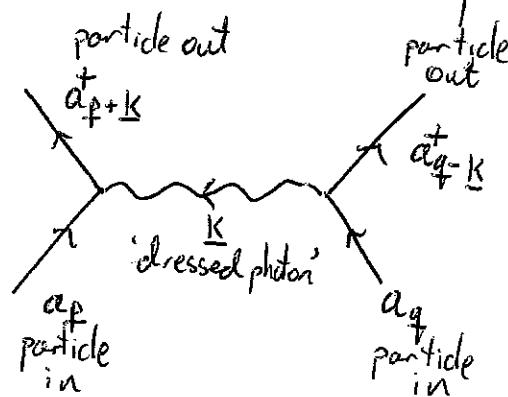
- Do the y integral to get $N S(\underline{g} + \underline{f} - \underline{q} - \underline{p})$ ie the interaction conserves momentum
- Use the S function to do the sum over $\underline{q} = \underline{p} + \underline{q} - \underline{f}$

$$\begin{aligned} \hat{V} &= \frac{1}{2N} \int d^3 z \sum_{\underline{p}, \underline{q}, \underline{f}} e^{-i(\underline{q}-\underline{f}) \cdot \underline{z}} V(\underline{z}) a_{\underline{p}+\underline{q}-\underline{f}}^+ a_f^+ a_q a_p \\ &= \frac{1}{2} \sum_{\underline{p}, \underline{q}, \underline{f}} \tilde{V}_{\underline{q}-\underline{f}} a_{\underline{p}+\underline{q}-\underline{f}}^+ a_f^+ a_q a_p \end{aligned}$$

Now for some interpretation: set $\underline{q} - \underline{f} = \underline{K}$

$$\hat{V} = \frac{1}{2} \sum_{\underline{p}, \underline{q}, \underline{K}} \tilde{V}_{\underline{K}} a_{\underline{p}+\underline{K}}^+ a_{\underline{q}-\underline{K}}^+ a_f^+ a_p$$

We can draw this in momentum space



The problem with this interaction is that it's generally impossible to solve for all but the simplest cases.

We need a scheme that approximates the interaction - there are several. These include:

- * Bogoliubov's hunting license \rightarrow superfluids (Bosons)
- * Hartree's tadpoles \rightarrow Metals
- * Fock's exchange oysters \rightarrow Metals & magnets
- * Cooper's instability \rightarrow Superconductors.

2 Famous examples

I The tight binding model of fermions on a lattice

$$\hat{H} = \sum_{ij} (-t_{ij}) \hat{c}_i^\dagger \hat{c}_j$$

kills a particle at j
 creates a particle at i
 reduce energy by t_{ij} for a hop $i \rightarrow j$

As usual we'll diagonalize by inserting a mode expansion

$$\hat{c}_i = \frac{1}{N} \sum_f \hat{c}_f e^{ip \cdot \underline{x}_i}$$

to obtain

$$\hat{H} = \frac{1}{N} \sum_{ij} \sum_{pq} (-t_{ij}) \hat{c}_q^\dagger \hat{c}_p \cancel{\hat{c}_q^\dagger \hat{c}_p} e^{-iq \cdot \underline{x}_i} e^{ip \cdot \underline{x}_j}$$

Make a change of variables $\underline{x}_i = \underline{x}_j + \underline{x}_\alpha$ and obtain

$$\hat{H} = \frac{1}{N} \sum_{\alpha j} \sum_{pq} (-t_\alpha) \hat{c}_q^\dagger \hat{c}_p e^{-iq \cdot (\underline{x}_j + \underline{x}_\alpha)} e^{ip \cdot \underline{x}_j}$$

Next we sum over ~~\underline{x}_j~~ to get $N S_{qp}$, then use the S-function (61)
to do a momentum sum

$$\hat{H} = \sum_\alpha \sum_p -t_\alpha \hat{c}_p^\dagger \hat{c}_p e^{-ip \cdot \underline{x}_\alpha} = \sum_p \left[\sum_\alpha (-t_\alpha) e^{-ip \cdot \underline{x}_\alpha} \right] \hat{c}_p^\dagger \hat{c}_p$$

which is diagonal with a dispersion given by $\epsilon_p = \sum_\alpha (-t_\alpha) e^{-ip \cdot \underline{x}_\alpha}$

For a cubic lattice with nearest-neighbour hopping we have

$$t_\alpha = t \quad \text{for } \underline{x}_\alpha = \pm \frac{1}{a} \underline{i}, \pm \frac{1}{a} \underline{j}, \pm \frac{1}{a} \underline{k}$$

$$= 0 \quad \text{for all other vectors}$$

$$\begin{aligned} \epsilon_p &= -t (e^{-ip_x a} + e^{ip_x a} + e^{-ip_y a} + e^{ip_y a} + e^{-ip_z a} + e^{ip_z a}) \\ &= -t (2 \cos p_x a + 2 \cos p_y a + 2 \cos p_z a) \end{aligned}$$

Famous examples II: The Hubbard model

(i) Consider a tight binding model with a potential

$$\hat{H} = \sum_{ij} (-t_{ij}) \hat{c}_i^\dagger \hat{c}_j + \frac{1}{2} \sum_{ijkl} V_{ijkl} \hat{c}_k \hat{c}_l$$

(ii) Now allow the electrons to carry a spin $\sigma = \uparrow$ or \downarrow . Hopping won't change the spin

$$\hat{H} = \sum_{ij\sigma} (-t_{ij}) \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + \frac{1}{2} \sum_{ijkl} V_{ijkl} \hat{c}_{k\sigma} \hat{c}_{l\sigma}$$

(iii) Hubbard then made the simplification $V_{ijkl} \Rightarrow V_{i\bar{i}i\bar{i}} = 2U$

$$\Rightarrow 0 \quad \text{for } i \neq j \neq k \neq l$$

$$\hat{H} = \sum_{ij\sigma} (-t_{ij}) \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + U \sum_i (\hat{c}_{i\uparrow}^\dagger \hat{c}_{i\uparrow})(\hat{c}_{i\downarrow}^\dagger \hat{c}_{i\downarrow})$$

or
$$\boxed{\hat{H} = \sum_{ij\sigma} (-t_{ij}) \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}}$$

Pauli prevents double occupancy with the same spin

This looks quite simple, but leads to a variety of complex and correlated ground states.

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Let's consider the 2 site version

$$t_{12} = t_{21} = t$$

site site basis states
1 2 $| \text{site 1, site 2} \rangle$

* If we put a single electron into the system we have a state

$$|\Psi\rangle = a|1,0\rangle + b|0,1\rangle \quad \text{and the Hamiltonian is written}$$

$$H = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} \quad \text{which has a ground state } |\Psi_0\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle)$$

* If we now put two electrons into the system with different spins then we ~~still~~ have states $|\Psi\rangle = a|1,0\rangle + b|1,1\rangle + c|0,1\rangle + d|0,0\rangle$ and a Hamiltonian

$$H = \begin{pmatrix} U & -t & -t & 0 \\ -t & 0 & 0 & -t \\ -t & 0 & 0 & -t \\ 0 & -t & -t & U \end{pmatrix}$$

The ground state of this Hamiltonian is $E = \frac{U}{2} - \frac{1}{2}(U^2 - 16t^2)^{\frac{1}{2}}$
 corresponding to a wavefunction

$$|\psi\rangle = N \begin{pmatrix} 1 \\ \frac{U}{4t} + \frac{1}{4t}[U^2 + 16t^2]^{\frac{1}{2}} \\ \frac{U}{4t} - \frac{1}{4t}[U^2 + 16t^2]^{\frac{1}{2}} \\ 1 \end{pmatrix}$$

with N a normalization constant

The message here is that U and t give rise to correlations in the form of complicated superpositions of the basis states.