

Elements of many body physics - Mean field theory

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We need a way to deal with the many-body potential $a^\dagger a^\dagger a a$.
There are many - we'll examine one here. It's a mean-field theory.

① We take the average of our operators

② We rewrite the average in terms of paired averages

Consider our unsolvable interaction

$$\hat{V} = \frac{1}{2} \sum_{f,k,q} V_q a_{f-q}^+ a_{k+q}^+ a_k^- a_f^-$$

Take the average in the form of a vacuum expectation value

$$\langle 0 | \hat{V} | 0 \rangle = \frac{1}{2} \sum V_q \langle 0 | a_{f-q}^+ a_{k+q}^+ a_k^- a_f^- | 0 \rangle$$

Next we express this in terms of pairs of operators using 'Wick's theorem' which says $\langle a_{f-q}^+ a_{k+q}^+ a_k^- a_f^- \rangle = \prod_{\text{all paired averages}}$

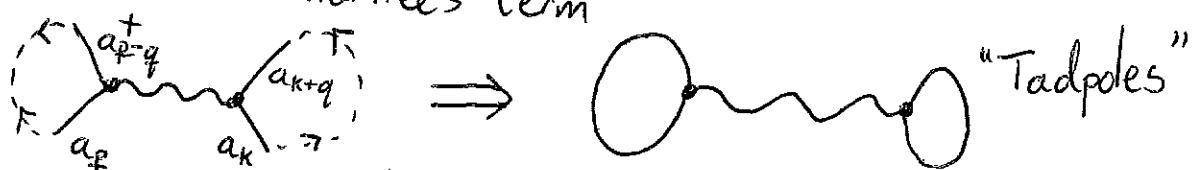
This gives us $\langle 0 | \hat{V} | 0 \rangle =$

$$\begin{aligned} & \langle 0 | a_{f-q}^+ a_{k+q}^+ | 0 \rangle \langle 0 | a_k^- a_f^- | 0 \rangle \\ & + \langle a_{f-q}^+ a_f^- \rangle \langle a_{k+q}^+ a_k^- \rangle \\ & - \langle a_{f-q}^+ a_k^- \rangle \langle a_{k+q}^+ a_f^- \rangle \\ & = C_0 + D_0 - E_0 \end{aligned}$$

What's the physics behind these averages?

C_0 = Cooper's term, it leads to superconductivity

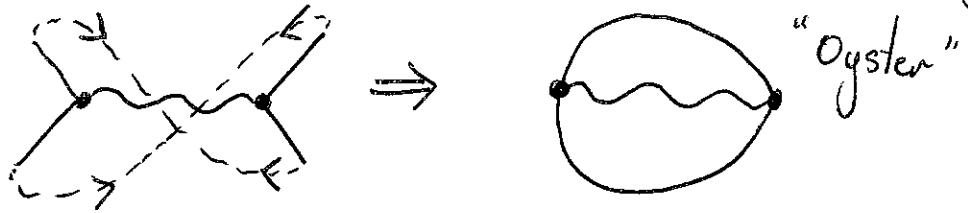
D_0 = Direct or Hartree's term



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E_0 = Exchange or Fock's term

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We'll ignore cooper's term for the moment. Our (rather sketchy) analysis suggests the ground state energy should have contributions from and .

$$\begin{aligned} \text{1st consider } \text{O}_q &= \frac{1}{2} \sum_{\substack{p \\ k}} V_q \langle a_{p-q}^+ a_p \rangle \langle a_{k+q}^+ a_k \rangle \\ &= \frac{1}{2} \sum_{\substack{p \\ k}} V_{q=0} \langle a_p^+ a_p \rangle \langle a_k^+ a_k \rangle \quad [\text{Need } p-q=p \text{ for } q \neq 0 \text{ result}] \\ &= \frac{1}{2} \sum_{p,k} V_{q=0} n_p n_k \end{aligned}$$

Taking a Fourier transform we can turn this into

$$\text{O}_q = \frac{1}{2} \int d^3x d^3y \langle \psi_{(x)}^+ \psi_{(x)} \rangle V_{(x-y)} \langle \psi_{(y)}^+ \psi_{(y)} \rangle$$

As you might guess for the energy of two charge distributions
What about the Fock term?

$$= -\frac{1}{2} \sum_{\substack{p \\ k \\ q}} V_q \langle a_{p-q}^+ a_k \rangle \langle a_{k+q}^+ a_p \rangle \quad [\text{Need } \begin{cases} p-q=k \\ k+q=p \end{cases} \text{ for } \neq 0]$$

$$\begin{aligned} &= -\frac{1}{2} \sum_{p,k} V_{p-k} \langle a_p^+ a_p \rangle \langle a_k^+ a_k \rangle \\ &= -\frac{1}{2} \sum_{p,k} V_{p-k} n_p n_k \\ &= -\frac{1}{2} \int d^3x d^3y \langle \psi_{(x)}^+ \psi_{(y)} \rangle V_{(x-y)} \langle \psi_{(y)}^+ \psi_{(x)} \rangle \end{aligned}$$

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The oyster is clearly far more complicated. Exchange is important in magnetism, as you may know! (69)

If we want to know about single particles we need only average two of the legs

$$= \sum_{\mathbf{k}} V_{q=0} n_{\mathbf{k}} \sum_{\mathbf{p}} a_{\mathbf{p}}^+ a_{\mathbf{p}}$$

$$= - \sum_{\mathbf{p}, \mathbf{k}} V_{q=-q} \langle a_{\mathbf{p}-\mathbf{q}}^+ a_{\mathbf{p}-\mathbf{q}} \rangle a_{\mathbf{p}}^+ a_{\mathbf{p}}$$

Since these are in the language of Feynman diagrams, you can use propagators to work out more complicated examples.

Here's a wonderful thing: we can sum the Hartree and Fock contributions to ALL ORDERS OF PERTURBATION THEORY! (70)

Here's how

$$\begin{aligned} \text{propagator} &= \uparrow + \overbrace{\uparrow \text{Hartree}}^{\text{Hartree}} + \overbrace{\uparrow \text{Fock}}^{\text{Fock}} + \overbrace{\uparrow \text{Hartree}}^{\text{Hartree}} + \overbrace{\uparrow \text{Fock}}^{\text{Fock}} + \dots \\ &= \frac{\uparrow}{1 - (\text{Hartree} + \text{Fock}) \times \uparrow} = \frac{1}{(\uparrow)^{-1} - (\text{Hartree} + \text{Fock})} \\ &= \frac{1}{E - E_{\mathbf{p}} - (\text{Hartree} + \text{Fock})} \end{aligned}$$

So the Hartree & Fock terms change the dispersion relations for the particles.

Superfluids & Bogoliubov's hunting license

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Consider a gas of Bosons with a hard-sphere potential

$$H = \sum_{\mathbf{p}} \frac{p^2}{2m} a_{\mathbf{p}}^+ a_{\mathbf{p}} + \frac{\lambda}{2} \sum_{\mathbf{p} \neq \mathbf{k}} a_{\mathbf{p}-\mathbf{q}}^+ a_{\mathbf{k}+\mathbf{q}}^+ a_{\mathbf{k}} a_{\mathbf{p}}$$

Bogoliubov's method is (again) to consider ground state averages of the operators. The difference is that he takes $\langle \Omega | a_{\mathbf{p}=0} | \Omega \rangle = \sqrt{N_0}$. What gives him license to do this?

If the n^o of particles in the ground state is very large (i.e. macroscopic) then $a_0 | \Omega \rangle = \sqrt{N_0 - 1} | \Omega \rangle \approx \sqrt{N_0} | \Omega \rangle$. The Hunting license allows us to replace operators with numbers. Of course we must have an even number of operators left, so we replace 2 or 4 operators.

Take pairs of momenta to zero to obtain

$$\textcircled{I} \quad \stackrel{\mathbf{p} \rightarrow 0}{k \rightarrow 0} a_{\mathbf{q}}^+ a_{\mathbf{q}}^+ a_0 a_0 = N_0 a_{-\mathbf{q}}^+ a_{\mathbf{q}}^+ \quad \textcircled{II} \quad \stackrel{\mathbf{p} \rightarrow 0}{\mathbf{p} - \mathbf{q} \rightarrow 0} a_0^+ a_{\mathbf{k}}^+ a_{\mathbf{k}} a_0 = N_0 a_{\mathbf{k}}^+ a_{\mathbf{k}}$$

$$\textcircled{III} \quad \stackrel{\mathbf{p} \rightarrow 0}{k+q \rightarrow 0} a_{-\mathbf{q}}^+ a_0^+ a_{-\mathbf{q}} a_0 = N_0 a_{-\mathbf{q}}^+ a_{\mathbf{q}} \quad \textcircled{IV} \quad \stackrel{k=0}{k+q=0} a_{\mathbf{p}}^+ a_0^+ a_0 a_{\mathbf{p}} = N_0 a_{\mathbf{p}}^+ a_{\mathbf{p}}$$

$$\textcircled{V} \quad \stackrel{k=0}{\mathbf{p} - \mathbf{q} \rightarrow 0} a_0 a_{\mathbf{q}}^+ a_0 a_{\mathbf{q}} = N_0 a_{\mathbf{q}}^+ a_{\mathbf{q}} \quad \textcircled{VI} \quad \stackrel{\mathbf{p} - \mathbf{q} = 0}{k+q=0} a_0^+ a_0^+ a_{-\mathbf{q}} a_{\mathbf{q}} = N_0 a_{-\mathbf{q}} a_{\mathbf{q}}$$

Replacing all subscripts gives N_0^2 .

We obtain the approximate interaction term

$$H_{\text{int}} = \frac{\lambda}{2} \left[N_0^2 + 4N_0 \sum_{\mathbf{p} \neq 0} a_{\mathbf{p}}^+ a_{\mathbf{p}} + N_0 \sum_{\mathbf{p} \neq 0} [a_{\mathbf{p}}^+ a_{-\mathbf{p}}^+ + a_{\mathbf{p}} a_{-\mathbf{p}}] \right]$$

But what is N_0 ? Particle number isn't conserved any more!

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To get round this we set $N = N_0 + \sum_{p \neq 0} a_p^+ a_p$

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Substitute this and we have the effective Hamiltonian

$$\hat{H} = \sum_{p \neq 0} \left(\frac{p^2}{2m} + \lambda N \right) a_p^+ a_p + \frac{1}{2} \sum_{p \neq 0} \lambda N (a_p^+ a_p^+ + a_p a_{-p})$$

where we've dropped the $\lambda N^2/2$ term and any operator combination that isn't bilinear.

So far, so good. EXCEPT $a^+ a^+$ and $a a$ aren't number operators. We may be in trouble! We need a way to turn these objects into things we can understand.

Luckily Bogoliubov also found the answer. Define new operators α and α^+ which destroy & create new particles - Bogolons.

$$\begin{aligned} \hat{\alpha}_p &= u_p \hat{a}_p - v_p \hat{a}_{-p}^+ \\ \hat{\alpha}_{-p}^+ &= -v_p \hat{a}_p + u_p \hat{a}_{-p}^+ \end{aligned} \quad \left. \begin{array}{l} \text{we can } \cancel{\text{express}} \\ \text{make a transformation} \\ \text{between the 2 sorts of operator} \end{array} \right\}$$

The new operators have the same commutation relations as the old ones $[\alpha_p, \alpha_q^+] = \delta_{pq}$ $[\alpha_p, \alpha_q] = [\alpha_p^+, \alpha_q^+] = 0$

For this to be true, we need $u_p^2 - v_p^2 = 1$, $u_p^+ = u_p$ and $v_p^+ = v_p$.

Nothing for it but to press on.

$$H = \sum_p (a_p^+ a_{-p}) \begin{pmatrix} \frac{p^2}{2m} & \frac{n\lambda}{2} \\ \frac{n\lambda}{2} & 0 \end{pmatrix} \begin{pmatrix} a_p \\ a_{-p}^+ \end{pmatrix}$$

We'll make the transformation $\begin{pmatrix} a_p \\ a_{-p}^+ \end{pmatrix} = \begin{pmatrix} u_p & -v_p \\ -v_p & u_p \end{pmatrix} \begin{pmatrix} \alpha_p \\ \alpha_{-p}^+ \end{pmatrix}$

and choose u_p and v_p such that

$$H = \begin{pmatrix} \alpha_p^+ & \alpha_{-p}^+ \end{pmatrix} \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix} \begin{pmatrix} \alpha_p \\ \alpha_{-p}^+ \end{pmatrix}$$

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$$\begin{aligned} \textcircled{1} \quad H &= \sum_p D_{11} \alpha_p^+ \alpha_p + D_{22} \alpha_{-p}^+ \alpha_{-p} \\ &= \sum_p D_{11} \alpha_p^+ \alpha_p + D_{22} (1 + \alpha_{-p}^+ \alpha_p) \end{aligned}$$

\textcircled{2} Reindex the second sum and the constant in front of $\alpha_p^+ \alpha_p$, which gives us the dispersion is $D_{11} + D_{22} = \text{Trace of the matrix.}$

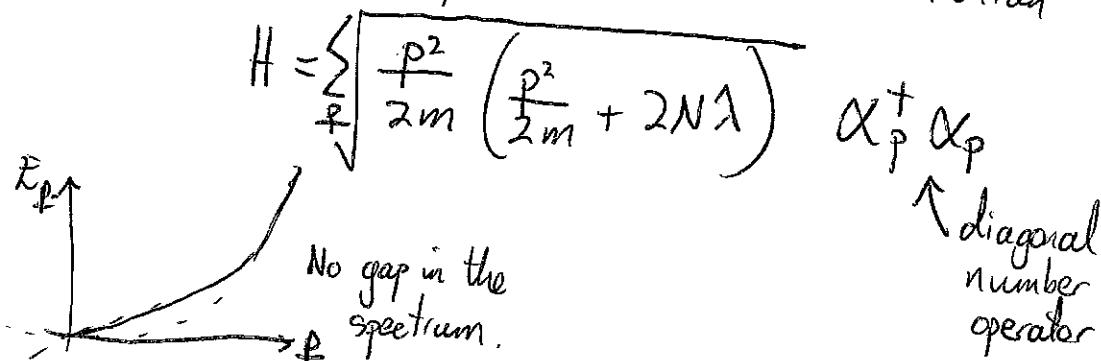
\textcircled{3} This trace is given by $\left(\frac{p^2}{2m} + N\lambda\right)(u_p^2 + v_p^2) - 2N\lambda u_p v_p = E_p$

\textcircled{4} We also want the off diagonal elements to be zero, meaning that we need

$$\frac{2u_p v_p}{u_p^2 + v_p^2} = \frac{N\lambda}{\frac{p^2}{2m} + N\lambda}$$

\textcircled{5} Eliminate u_p and v_p

We obtain the dispersion relation Hamiltonian



Success! We've diagonalized the Hamiltonian by replacing a^+ and a with α^+ and α .

Bogolons have a dispersion relation given by $E_p = \left(\frac{N\lambda}{m}\right)^{\frac{1}{2}} |p|$ at low momenta and $E_p = \frac{p^2}{2m}$ at large momenta. Bogolons are the single particle excitations in a superfluid.

Perturbation theory

There aren't many solvable problems in QFT. Most results rely on some approximate methods. We'll discuss some of these here.

There are (very) many books on this subject - see the reading list.

I What are we trying to work out?

We've done away with single particle wavefunctions (we turned them into operator valued fields) so it's not obvious what to calculate to get the important information from a system.

The answer turns out to be that all of the information is contained in an amplitude called the PROPAGATOR (~~also~~ also known as the Kernel or the Green's function). It is defined as the amplitude

$$G(x,y) = \left\langle \begin{array}{c|c} \text{Test particle} & \text{Test particle starts} \\ \text{ends up at position } & \text{at position } y \text{ at } \\ \underline{x} \text{ at time } x^0 & \text{time } y^0 \end{array} \right\rangle$$

So all perturbation theory relies on doing a single experiment

(I) Take a complicated system in its ground state $|1\rangle$

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(II) Add a particle at y at time $t=y^0$ using the operator $\hat{\phi}^+(y)$

(III) Remove the particle at x at time x^0 with an operator $\hat{\phi}(x)$

(IV) Find the amplitude that the particle is still in its groundstate by hitting it with $\langle 2 |$

We can write the experiment as

$$G(x,y) = \langle 2 | T \hat{\phi}(x) \hat{\phi}(y) | 1 \rangle$$

here T is ~~a~~ a symbol that ensures we create the particle ~~before~~ before we annihilate it.

We define the time ordering symbol via

$$T \hat{A} \hat{B} = \begin{cases} \hat{A}(t_2) \hat{B}(t_1) & \text{if } t_2 > t_1 \\ \hat{B}(t_1) \hat{A}(t_2) & \text{if } t_1 > t_2 \end{cases}$$

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So the propagator is

$$G = \langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \Theta(x^0 - y^0) \langle \Omega | \phi(x) \phi(y) | \Omega \rangle + \Theta(y^0 - x^0) \langle \Omega | \phi(y) \phi(x) | \Omega \rangle$$

Let's examine the propagator further in a scalar field theory

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 + \dots \quad \text{with } \phi = \phi^+, \text{ ground state } |\Omega\rangle$$

This is ϕ^4 theory and is unsolvable. We'll treat ϕ^4 as a perturbation. We'll start off by asking about what happens when $\lambda = 0$. This is then a free (solvable) theory whose ground state is called $|0\rangle$

We can find the propagator for the free theory

$$G_0 = \langle 0 | T \phi(x) \phi(y) | 0 \rangle = \Theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \Theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle$$

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This is known as a "Feynman propagator". By inserting the mode expansion we find

$$G(x, y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{(2E_p)} \left[\Theta(x^0 - y^0) e^{-ip(x-y)} + \Theta(y^0 - x^0) e^{-ip(y-x)} \right]$$

which can be thought of as

$$G_0(x, y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left[\underbrace{\left(\text{Particle goes from } y \text{ to } x \right)}_{\text{from } y \text{ to } x} + \underbrace{\left(\text{Antiparticle goes from } x \text{ to } y \right)}_{\text{from } x \text{ to } y} \right]$$

By using the representation $\Theta(x) = \int \frac{dz}{2\pi} \frac{e^{-izx}}{z+iE}$ we can get rid of the Θ functions and obtain (81)

$$G_0(x,y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2 - m^2 + iE}$$

which is the Fourier transform of $\tilde{\Delta}(p) = \frac{i}{p^2 - m^2 + iE}$ ← momentum space free propagator for the particle

So far, so good. Next we need to know how to make a perturbation theory using propagators.

To do this we exploit some 19th century mathematics. $\Delta(x,y)$ is the Green's function of the equation of motion for the field. That is to say $(\partial^2 + m^2) G_0(x,y) = -i \delta(x-y)$

Green's functions

It turns out that $(\partial^2 + m^2) G_0(x,y) = -i \delta(x-y)$ ①

We want to find the more general propagator when we have an extra term in the equation of motion

$$(\partial^2 + m^2 + V) G = -i \delta(x-y) \quad ②$$

Written symbolically in momentum space we have

$$(\partial^2 + m^2) G_0 = -i \delta$$

$$(-p^2 + m^2) G_0 = -i$$

$$G_0 = \frac{i}{p^2 - m^2} \quad \text{as before}$$

and by the same token

$$\tilde{G} = \frac{i}{p^2 - m^2 - V}$$

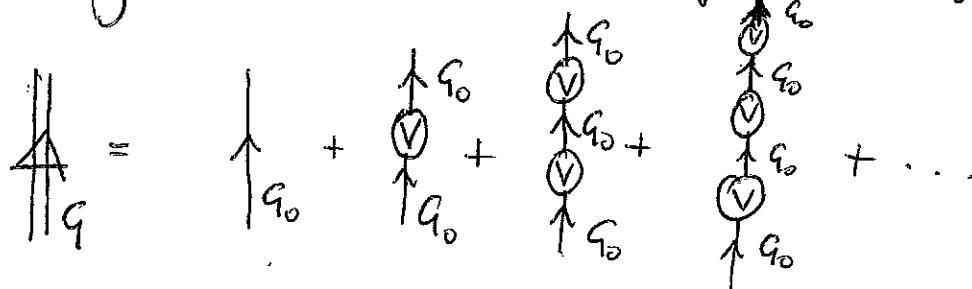
This is useful because of a matrix identity which says (83)

$$\frac{1}{A+B} = \frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} + \dots$$

so $G = \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2} V \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2} V \frac{i}{p^2 - m^2} V \frac{i}{p^2 - m^2} + \dots$

$$= G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \dots$$

This may be drawn in cartoon form as follows



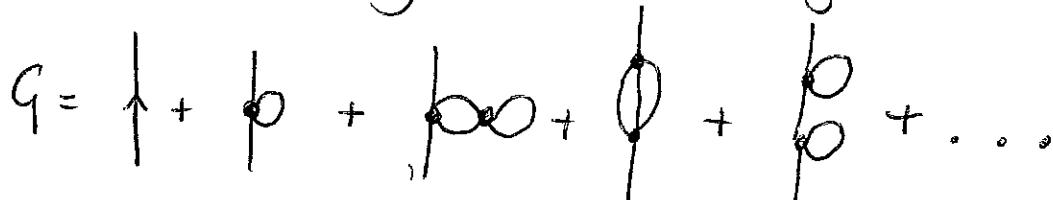
This is a perturbation series. We interpret this as a free particle propagating until it interacts with \textcircled{V} , after which it propagates freely again, and so on.

Of course, the difficulty lies in untangling what goes into the \textcircled{V} blob, but the philosophy turns out to be sound. (84)

Feynman sorted out the details ~~in~~ and came up with a system for doing these calculations. He turned all of the terms in the perturbation series into diagrams and then wrote down the answer

$G = (\text{sum of all Feynman diagrams with two external legs})$

For the $\lambda\phi^4$ theory we end up having to calculate



We'll look at these in more detail next.

Feynman diagrams

Student: Why aren't we going to learn this Feynman's way?

Gell-Mann: Feynman uses a different method to us

Student: What's Feynman's method?

Gell-Mann: Feynman writes down the question, thinks about it, then writes down the answer.

We're going to use Feynman's method to do perturbation theory. Feynman says:

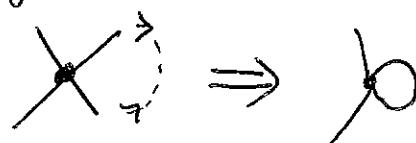
$$G^{(2)} = \sum \left(\begin{array}{l} \text{All Feynman diagrams} \\ \text{with 2 external legs} \end{array} \right)$$

We'll do this for ϕ^4 theory.

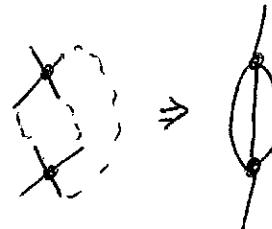
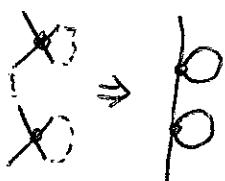
Remember that we had $\mathcal{L} = \frac{1}{2}(\partial\mu\phi)^2 - \frac{m^2}{2}\phi^4 - \frac{1}{4!}\lambda\phi^4$. For $\lambda=0$ the theory is solvable, for $\lambda \neq 0$ we treat the term $-\frac{\lambda}{4!}\phi^4$ as a perturbation.

Feynman says that we draw the 'interaction part' (the perturbation) like ~~(-i)~~ with 4 lines corresponding to 4 copies of ϕ in $-\frac{\lambda}{4!}\phi^4$ we then have to join up lines in all of the possible ways to get diagrams with 2 external lines.

Example Diagrams with 1 interaction vertex



Example Diagrams with 2 interaction vertices



We then need a set of rules for working out an amplitude corresponding to each diagram. The rules go like

- ① Each line contributes a free propagator $\frac{i}{p^2 - m^2 + i\epsilon}$
- ② Each interaction vertex contributes a factor $\times = (-i\lambda)$
- ③ Conserve momentum at the vertices
- ④ Divide by a combinatoric symmetry factor *
- ⑤ Integrate over all intermediate momenta $\int \frac{d^4 q}{(2\pi)^4}$

Examples

$$Q_k \propto \frac{i}{p^2 - m^2 + i\epsilon} \left[\int \frac{d^4 k}{(2\pi)^4} \frac{i(-i\lambda)}{k^2 - m^2 + i\epsilon} \right] \frac{i}{p^2 - m^2 + i\epsilon}$$


* we'll discuss this one separately