

Path Integrals - another way to do quantum mechanics

(88)

We've found that you often want to know the amplitude for a process like

$$G = \langle \underset{\text{at time } y^0}{\text{end up at } y} | \underset{\text{at time } x^0}{\text{start at } x} \rangle = \begin{array}{c} y = (y, g) \\ x = (x^0, x) \end{array}$$

Classically you find the path from $y \rightarrow x$ by minimizing the action

$S = \int L[q(t)] dt$ which gives you the trajectory that the particle takes: (usually) the shortest one.

Richard Feynman's method is to calculate $G = \sum e^{iS/\hbar}$ where the sum is over the action for every single possible trajectory a particle can take in getting from y to x .



Add up all of the paths & you get G .

Deriving the integral

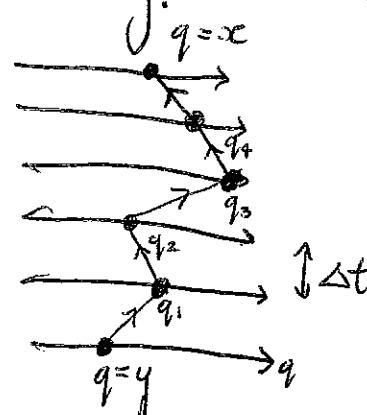
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$$G = \langle x | \hat{U}(x^0, y^0) | y \rangle = \langle x | e^{-i\hat{H}(x^0 - y^0)} | y \rangle$$

① Split this into N smaller steps

$$G = \langle x | (e^{-i\hat{H}\Delta t})^N | y \rangle = \langle x | \underbrace{e^{-i\hat{H}\Delta t} \dots e^{-i\hat{H}\Delta t}}_{N \text{ copies.}} \dots e^{-i\hat{H}\Delta t} | y \rangle$$

Each evolution operator moves a particle between 2 closely spaced positions q_n and q_{n+1} , say. We've essentially sliced up time



Next we insert a resolution of the identity $\{dq_n | q_n \rangle \langle q_n|\} = 1$ in between each propagator mini time-evolution operator

$$G = \langle x | e^{-iH\Delta t} \left[\{dq_{N-1} | q_{N-1} \rangle \langle q_{N-1}|\} \right] e^{-iH\Delta t} \dots e^{-iH\Delta t} \left[\{dq_{n+1} | q_{n+1} \rangle \langle q_{n+1}|\} \right] \times \\ \times e^{-iH\Delta t} \left[\{dq_n | q_n \rangle \langle q_n|\} \right] e^{-iH\Delta t} \dots e^{-iH\Delta t} \left[\{dq_1 | q_1 \rangle \langle q_1|\} \right] e^{-iH\Delta t} | y \rangle$$

Each resolution contains an integral over q_i . We obtain a new trajectory for each value of q_i we integrate over. A little rearranging gives us

$$G = \int dq_1 \dots dq_{N-1} \langle y | e^{-iH\Delta t} | q_{N-1} \rangle \dots \langle q_{n+1} | e^{-iH\Delta t} | q_n \rangle \dots \\ \dots \langle q_1 | e^{-iH\Delta t} | y \rangle$$

So the amplitude contains a string of mini propagators

$$G_n = \langle q_{n+1} | e^{-iH\Delta t} | q_n \rangle = \langle q_{n+1} | e^{-i(\frac{\hat{p}^2}{2m} + \hat{V}(q))\Delta t} | q_n \rangle$$

Evaluating a minipropagator

$$G_n = \langle q_{n+1} | e^{-i(\frac{\hat{p}^2}{2m}\Delta t + \hat{V}(q))\Delta t} | q_n \rangle \quad (91) \\ = \langle q_{n+1} | e^{-i\frac{\hat{p}^2}{2m}\Delta t} | q_n \rangle e^{-iV(q)\Delta t} \\ = \langle q_{n+1} | e^{-i\frac{\hat{p}^2}{2m}\Delta t} \left[\{dp | p \rangle \langle p | q_n\} \right] e^{-iV(q)\Delta t} \\ = \int dp \langle q_{n+1} | p \rangle e^{-i\frac{\hat{p}^2}{2m}\Delta t} \langle p | q_n \rangle e^{-iV(q)\Delta t} \\ = \int \frac{dp}{(2\pi)} e^{ipq_{n+1}} e^{-i\frac{\hat{p}^2}{2m}\Delta t} e^{-ipq_n} e^{-iV(q)\Delta t} \\ = \int \frac{dp}{(2\pi)} e^{-i\frac{p^2}{2m}\Delta t + ip(q_{n+1} - q_n)} e^{-iV(q)\Delta t}$$

This is an integral we can do. It's based on the identity

$$\int dx e^{-\frac{ax^2}{2} + bx} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} e^{\frac{b^2}{2a}}$$

With $a = \frac{i\Delta t}{m}$ and $b = i(q_{n+1} - q_n)$ we have

$$G_n = \left(\frac{-2\pi im}{2\pi\Delta t} \right)^{\frac{1}{2}} e^{\frac{+im(q_{n+1}-q_n)^2}{2(\Delta t)^2}\Delta t} e^{-iV(q)\Delta t}$$

Finally we can build up G from these operators

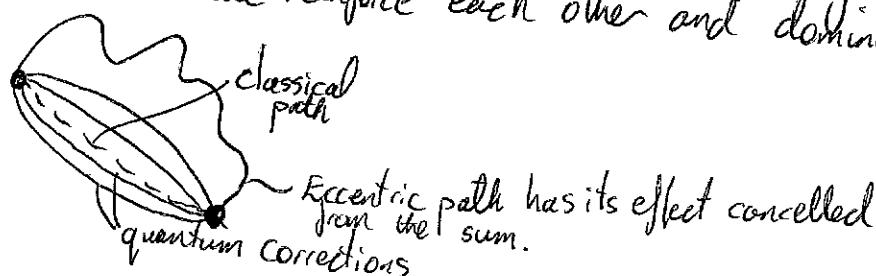
$$G = \prod_{n=1}^{N-1} \int dq_n \left(\frac{-2\pi im}{2\pi\Delta t} \right)^{\frac{N-1}{2}} e^{\frac{im}{2} \sum_n \frac{(q_{n+1}-q_n)^2}{(\Delta t)^2} \Delta t} e^{-iV(q)\Delta t}$$

we take the limit $N \rightarrow \infty, \Delta t \rightarrow 0$ which gives $\frac{q_{n+1}-q_n}{\Delta t} \rightarrow \dot{q}$
and $\sum_n \Delta t \rightarrow \int dt$ and we obtain

$$G = \int D[q(t)] e^{i \int dt L[q(t)]}$$

$$\text{where } D[q(t)] = \prod_{n=1}^{N-1} \int dq_n \left(\frac{-2\pi mi}{2\pi\Delta t} \right)^{\frac{N-1}{2}} \text{ and } L[q(t)] = \frac{m\dot{q}^2}{2} - V(q)$$

The answer is $G = \int D[q] e^{iS/\hbar}$ when we restore t 's and recognise the action $S = \int dt L[q(t)]$. We therefore add up all of the trajectories to get the amplitude. However, not all paths contribute equally. Paths close to the classical value reinforce each other and dominate the sum.



We can do the same calculation for fields and we obtain

$$G = \int D[\phi(x)] e^{i \int l(\phi) dx}$$

↑ integral over all field configurations. ↑ Lagrange density

The Renormalization group

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One question we keep asking in physics is 'When is a theory valid?'

A problem that nearly killed off QFT is the following

$$\text{Diagram: } q = p_1 + p_2 - p \quad L = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4$$
$$\text{Integration path: } \int_0^\infty \frac{d^4 p}{q^2 - m^2} \quad \frac{1}{p^2 - m^2}$$

λ has ~~4~~ powers of momentum on top and ~~4~~ on the bottom. At large p it goes as $\int \frac{d^4 p}{p^4}$. This is like $\int_a^\infty \frac{dx}{x} = \ln(\infty/a)$. This divergent amplitude is a serious problem.

The solution is to integrate up to a large momentum Λ rather than ∞ . As long as $\Lambda \gg$ the largest momentum in our problem then all is well.

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However Λ is an arbitrary constant - we don't want it to appear in anything that we measure.

* The first solution to this problem is known as 'renormalization'. It involves cancelling Λ from equations by changing coupling constants (like m and λ) sometimes by infinite amounts. The coupling constants change with momentum scale, so the physics is different on different energy scales.

Note that large $p =$ short distance so we can talk about Λ being the smallest distance we want to consider.

In QM $\Lambda \approx$ size of an atom for most applications

The RNG is a method to examine how physics varies with Λ .
 Specifically it tells us how the coupling constants change as we look at a system at different energy/length scales.

The method is quite simple: you just perform your path integrals a little bit at a time.

Consider a theory $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \sum_i g_i \phi^i$

The amplitude for starting with a vacuum and ending with a vacuum is given by

$$Z = \int \mathcal{D}[\phi] e^{-\int L d^d x}$$

dimensionality
of the problem
integrate up to L only

To implement the RNG we only integrate over the largest momenta i.e. those between bL and L (where $b < 1$). This will remove the very wiggly parts of our fields and leave the slowly varying parts untouched.

$$\phi = \underbrace{\dots}_{= \phi_s + \phi_f} \rightarrow \phi' = \underbrace{\dots}_{}$$

To do this we divide the integral into two parts

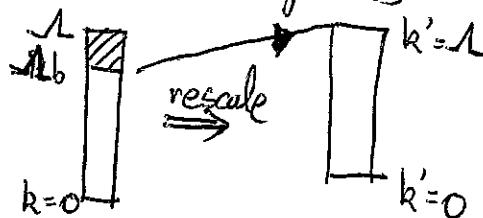
$$\begin{aligned} Z &= \int_0^{bL} \mathcal{D}\phi_s e^{-\int L(\phi_s)} \times \int_{bL}^L \mathcal{D}(\phi_s, \phi_f) e^{-\int L(\phi_s, \phi_f)} \\ &= \int_0^{bL} \mathcal{D}\phi_s e^{-\int L(\phi_s)} \times e^{-\int \delta L \phi_s} \quad \downarrow \text{Do this integral to get} \\ &= \int_0^{bL} \mathcal{D}\phi_s e^{-\int L(\phi_s) + \delta L(\phi_s)} \end{aligned}$$

Doing the integral is the hard part, and often involves an approximation

The answer should be of the same FORM as before, but will have different coefficients.

$$\mathcal{L} + S_d = \frac{1}{2} (\partial_\mu \phi_s)^2 + \sum_n g_n \phi_s^n$$

We'll compare this to what we started with. To do this we need to RESCALE the fields



$$k' = k/b \text{ and } x' = xb$$

$$\int \mathcal{L}(\phi_s) + \dots \text{ becomes } \int d^d x b^{-d} \left[\frac{b^2}{2} (\partial_{\mu} \phi_s')^2 + \sum_n g'_n \phi_s'^n \right]$$

Lastly rescale the field so that the first term looks like it did before

$$\int d^d x \left[\frac{1}{2} (\partial_{\mu} \phi_s')^2 + \sum_i b^{-d+\frac{n}{2}(d-2)} g'_n (\phi_s')^n \right]$$

This rescaling changes the coupling constants

$$g'_n = b^{\frac{n}{2}(d-2)-d} g_n$$

$$\text{Suppose } d=4 \quad g'_n = b^{n-4} g_n \quad n=2 \quad (g_2 \equiv m)$$

$$g'_2 = b^{-2} g_2$$

$b < 1$, so g'_2 gets larger

$$n=6$$

$$g'_6 = b^2 g_6$$

so g'_6 gets smaller

This tells us which coupling constants are important in the theory.

So much for the method. What's the result?

The end result is a law for the coupling constant.

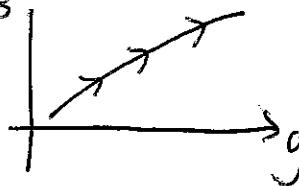
$$\frac{dg}{dL} = \beta(g)$$

length scale $\Delta L > 0$ in the procedure.

Reality corresponds to sending $L \rightarrow \infty$, (looking at the longest length scales).

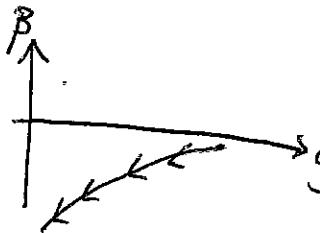
Some examples

(i)

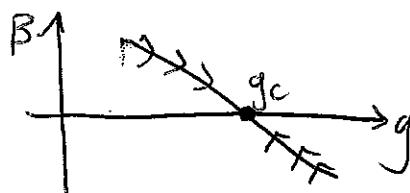


$$\begin{aligned}\beta &\text{ +ve} \\ \Delta g &= \beta \Delta L \\ g &\rightarrow \infty\end{aligned}$$

(ii)

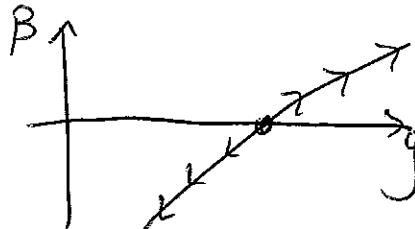


$$\begin{aligned}\beta &\text{ -ve} \\ \Delta g &= \beta \Delta L \\ g &\rightarrow 0\end{aligned}$$



$\beta = 0$ at $g = g_c$
At this point $\frac{dg}{dL} = 0$ and the flow stops

The system flows towards the fixed point.

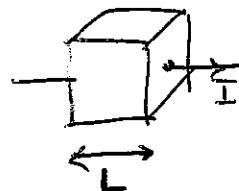


This is a repulsive fixed point. It dominates the flow, but the system never reaches it.

Example from CMP Anderson localization

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In 3D

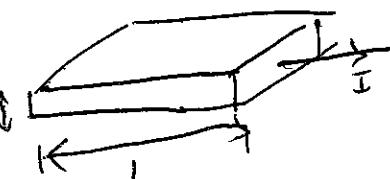


$$I = JL^2 \\ = \sigma EL^2 \\ = \sigma VL$$

Conductance

$$G = \frac{I}{V} = \sigma L$$

In 2D



$$I = JaL \\ = \sigma E aL \\ = \sigma V a \\ G = \frac{I}{V} = \sigma a$$

$$\text{In general } G(L) \propto L^{D-2}$$

For an insulator, we expect $G(L) \approx ce^{-L/\xi}$

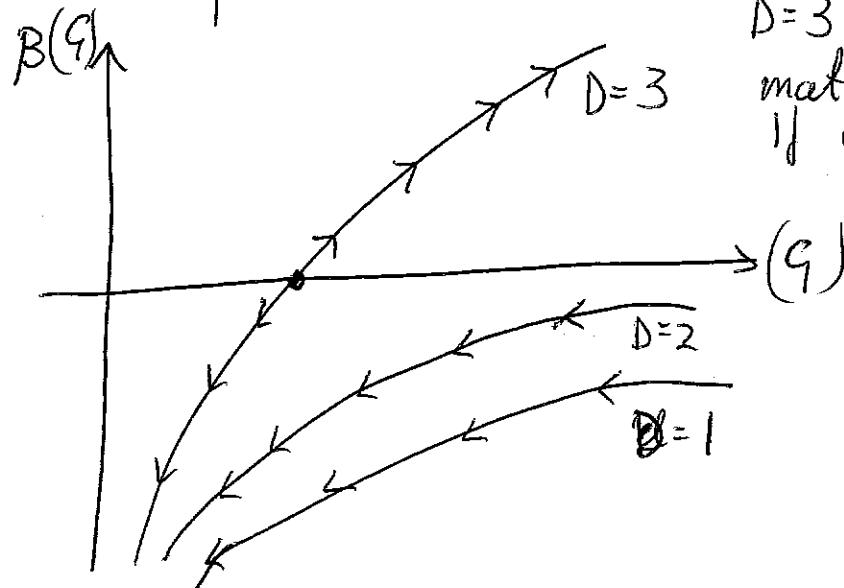
$$L \times \frac{dG}{dL} = -\frac{L}{\xi} G(L) = G(L) [\ln G(L) - \ln c]$$

Conclusion $\beta(G) = \frac{L}{G} \frac{dG}{dL} = \begin{cases} D-2 & \text{large } G \\ \ln g & \text{small } g \end{cases}$

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We didn't even need to do an integral!

What do we expect



$D=3$ If $G > G_c$ the material conducts.
If $G < 0$ it's an insulator.

$D=2, 1$ $G \rightarrow 0$

All states are localized and the materials are insulators