Simple Harmonic Oscillator

The simple harmonic oscillator problem is described by the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2\hat{x}^2}{2}$ and the commutation relation $[x, p] = \mathrm{i}\hbar$. Consider the creation ($\hat{a}^\dagger$) and annihilation ($\hat{a}$) operators we defined in the lecture.

1. Show that $[\hat{a}, \hat{a}] = 0$, $[\hat{a}^\dagger, \hat{a}^\dagger] = 0$, $[\hat{a}, \hat{a}^\dagger] = 1$ and $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$.

2. Consider a perturbation to this Hamiltonian $\beta\hat{x}^3 + \gamma\hat{x}^4$ where β and γ are small. By writing the perturbation in terms of creation and annihilation operators of the original Hamiltonian, show that the first-order shift in the ground-state energy of the system, due to these anharmonic parts, is given by
$$\Delta E = \frac{3}{4}\gamma\left(\frac{\hbar}{m\omega}\right)^2.$$

3. (a) Show that the transformation $\hat{b} = u\hat{a} + v\hat{a}^\dagger$ and $\hat{b}^\dagger = u\hat{a}^\dagger + v\hat{a}$ (with u and v real), preserves the commutation relations, as long as $u^2 - v^2 = 1$.

   (b) Using the results of (a), diagonalize the Hamiltonian
$$H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \frac{\Delta}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}\right), \tag{1}$$
   by transforming it into the form $H = \hbar\varepsilon\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right)$ and find ε. This is an example of a Bogoliubov transformation and is a useful trick to diagonalize a Hamiltonian. *Hint: If you have a problem with the algebra, see J.F. Annett, Superconductivity, Superfluids and Condensates for some help.*

Quantum fields

1. (a) Show that $\mathrm{i}\frac{\partial\hat{U}}{\partial t} = \hat{H}\hat{U}$, where $\hat{U}$ is shorthand for the time evolution operator $\hat{U}(t, 0) = e^{-\mathrm{i}\hat{H}t}$.
   (b) By differentiating $\hat{O}_{\mathrm{H}}(t) = e^{\mathrm{i}\hat{H}t}\hat{O}e^{-\mathrm{i}\hat{H}t}$, derive Heisenberg's equation of motion.

2. (a) $\hat{V}(\mathbf{a})$ is a translation operator with the property $\hat{V}(\mathbf{a})|\mathbf{x}\rangle = |\mathbf{x} + \mathbf{a}\rangle$. Show that, for an operator valued field $\hat{\phi}(\mathbf{x})$, we have $\hat{V}^\dagger(\mathbf{a})\hat{\phi}(\mathbf{x})\hat{V}(\mathbf{a}) = \hat{\phi}(\mathbf{x} - \mathbf{a})$.

   (b) By considering an infinitesimal translation show that an explicit form for the translation operator is $\hat{V}(\mathbf{a}) = e^{-\mathrm{i}\hat{\mathbf{p}}\cdot\mathbf{a}}$. *Hint: If you find you have the wrong sign in your exponential, consider the difference between translating a particle and allowing it to evolve.*

3. In this problem we'll canonically quantize a system described by the complex scalar field Lagrangian
$$\mathcal{L} = \partial_\mu\psi^\dagger(x)\partial^\mu\psi(x) - m^2\psi^\dagger(x)\psi(x). \tag{2}$$
   Note that in this system we can treat the ψ and $\psi^\dagger$ fields as independent. (It's an interesting questions to consider why you're allowed to do this!)

   (a) Show that the momentum density conjugate to the ψ field is $\Pi^0_\psi = \partial^0\psi^\dagger$ and find $\Pi^0_{\psi^\dagger}$.

   (b) Find the Hamiltonian density $\mathcal{H} = \sum_a \Pi^0_a\partial_0\psi^a - \mathcal{L}$.

(c) The mode expansion to use here is

$$\hat{\psi}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-\mathrm{i}p\cdot x} + \hat{b}^{\dagger}_{\mathbf{p}} e^{\mathrm{i}p\cdot x} \right) \tag{3}$$

with equal time commutation relations $\left[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}\right] = \left[\hat{b}_{\mathbf{p}}, \hat{b}^{\dagger}_{\mathbf{q}}\right] = \delta^{(3)}(\mathbf{p}-\mathbf{q})$. Insert the mode expansion and show that the normal ordered Hamiltonian is given by

$$\hat{H} = \int \mathrm{d}^3 p E_{\mathbf{p}} \left(\hat{a}^{\dagger}_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \hat{b}^{\dagger}_{\mathbf{p}} \hat{b}_{\mathbf{p}} \right), \tag{4}$$

and interpret this result. *For help see Aitchison and Hey.*

Examples of second quantization

1. An electron system has three momentum states, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, and is described by a Hamiltonian

$$\hat{H} = E_0 \sum_{\mathbf{p}} \hat{d}^{\dagger}_{\mathbf{p}} \hat{d}_{\mathbf{p}} - \frac{V}{2} \sum_{\mathbf{pk}} \hat{d}^{\dagger}_{\mathbf{k}} \hat{d}_{\mathbf{p}}. \tag{5}$$

States are expressed using a basis $|n_{\mathbf{p}_1} n_{\mathbf{p}_2} n_{\mathbf{p}_3}\rangle$ and if we put a single electron into the system then its state may be written $|\psi\rangle = a|100\rangle + b|010\rangle + c|001\rangle$.

Show that the Hamiltonian takes the form

$$\hat{H} = \left[E_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{V}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right]. \tag{6}$$

Find the energy eigenvalues and the corresponding eigenstates.

2. The nearest neighbour Hubbard model Hamiltonian may be written

$$\hat{H} = -t \sum_{\langle ij \rangle} (\hat{c}^{\dagger}_{i\sigma} \hat{c}_{j\sigma} + \hat{c}^{\dagger}_{j\sigma} \hat{c}_{i\sigma}) + U \sum_{i} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}, \tag{7}$$

where the first sum is over unique nearest neighbours. Consider a system with two possible sites for electrons.

(a) Put a single electron in the system. Using a basis $|\uparrow, 0\rangle$ and $|0, \uparrow\rangle$ show that the Hamitonian is given by

$$\hat{H} = \begin{pmatrix} 0 & -t \\ -t & 0 \end{pmatrix}. \tag{8}$$

Find the energy eigenvalues and eigenstates.

(b) Now put a second electron into the system with opposite spin to the first. Now using the basis states $|\uparrow\downarrow, 0\rangle$; $|\uparrow, \downarrow\rangle$; $|\downarrow, \uparrow\rangle$; and $|0, \downarrow\uparrow\rangle$, show that the Hamiltonian becomes

$$\hat{H} = \begin{pmatrix} U & -t & -t & 0 \\ -t & 0 & 0 & -t \\ -t & 0 & 0 & -t \\ 0 & -t & -t & U \end{pmatrix}. \tag{9}$$

Diagonalize this to obtain the eigenstates and energy eigenvalues. *Hint: There's no shame in using a computer if you like!*

Propagators and perturbation theory

1. Show that the single particle propagator $G = \langle x, t_x | y, t_y \rangle$ may be written

$$G = \sum_p \phi_p(x)\phi_p^*(y)e^{-\mathrm{i}E_p(t_x - t_y)}. \tag{10}$$

2. Prove the most important result in the path integral version of quantum field theory:

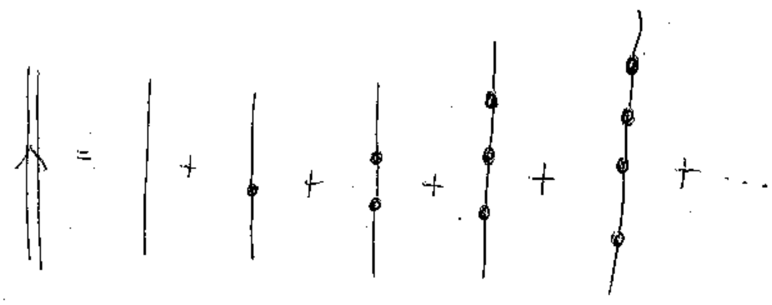
$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{ax^2}{2}+bx} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}. \tag{11}$$

3. For non-relativistic, free particles, show that the propagator is given by

$$G = \sqrt{\frac{m}{2\pi\mathrm{i}(t_x - t_y)}} e^{\frac{\mathrm{i}m(x-y)^2}{2(t_x - t_y)}}. \tag{12}$$

4. Consider the Lagrangian density $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2$. We're going to treat the mass term as a perturbation by splitting the theory into a free part $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2$ and an interacting part $\mathcal{L}_{\text{int}} = -\frac{m^2}{2}\phi^2$. The free propagator is given, in momentum space, by $G_0(p) = \frac{\mathrm{i}}{p^2}$.

   In order to see how the perturbation modifies the propagator consider the infinite sum of diagrams in the figure.



   If each interaction blob contributes a factor $-\mathrm{i}m^2$ show that the full propagator is given by

$$G = \frac{\mathrm{i}}{p^2 - m^2}. \tag{13}$$